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JOURNAL OF THE AERONAUTICAL SCIENCES

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Progress in Dynamic Stability and Control Research

WILLIAM F. MILLIKEN, JR.*

Cornell Aeronautical Laboratory

SUMMARY

This report presents a discussion of an extensive flight and analytical research program in dynamic stability and control which is being performed at the Cornell Aeronautical Laboratory. The basic objective of this program is that of bringing dynamic analysis closer to the needs of the airplane, missile, and automatic control designer, through a simplification of approach and by the development of methods for obtaining quantitative dynamic information in flight. Automatic control, in the form of modified automatic pilots, is used throughout the flight program.

In Part I of this report a review of the significance, background, and scope of the work is given. The important technical aspects are then discussed in Parts II and III, which survey the work done in the longitudinal motion over the last 2½ years, while in Part IV the future direction of the overall program is outlined. Finally, an opportunity is taken to summarize in the Appendix other dynamic research, both as a background for the present efforts and because of its inherent interest at the present time.

The program to date has resulted in the development of a number of flight and analytical methods for the measurement and handling of dynamic phenomena. Some of these are immediately applicable to design problems. The most outstanding accomplishment of the program, from an overall point of view, has been its influence in furthering the use of control forcing functions (and automatic control as a means of applying them) in the determination of aerodynamic information.

INTRODUCTION

THERE ARE CERTAIN PROBLEMS in the design of airplanes and other types of flying devices which depend for solution upon a detailed knowledge of the transient velocity conditions that result from flight control use and atmospheric or other disturbance. Associated with the time history of the motion—unsusceptible to rigorous solutions by static analysis alone—these problems come under the broad classification of dynamic stability and control.

Rapid developments in missiles and conventional aircraft have placed a premium on new methods and techniques that will enable the potential utility of dynamic analysis to be realized in design.

The present report reviews the salient features of a full-scale dynamic stability and control research program now in progress at the Cornell Aeronautical Laboratory and its relation to allied investigations. During this program extensive measurements are being made of airplane response to control motions that are applied by means of automatic control.

In the first part of this report an examination is made of the general background and objectives of the work, while the remaining parts are devoted to specific phases of the program.

PART I—GENERAL

The Necessity for Dynamic Research

While, from the earliest days of the airplane, it was appreciated that a complete understanding of stability and control involved dynamic analysis (taking into account not only the static attitude restoring terms but all of the aerodynamic and inertia effects arising from the motion), relatively little was accomplished in the ensuing years to make this a commonly accepted tool in design. In general, little was done toward simplification of the mathematical approach, determination of the operating limits within which the classical assumptions were valid, or devising methods of obtaining specific experimental data (either from wind-tunnel or flight test) for use in design calculations and full-scale changes and improvements. This lack of engineering adoption is, in part, explainable by the fact that it was possible to develop satisfactory airplanes on the basis of evolutionary design experience, static wind-tunnel test, and cut-and-try modification stemming from flight tests on the prototype and service use on subsequent

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* Flight Research Department.

articles, while, in addition, the existing specifications at that stage of the airplane art did not, in most cases, warrant the cost of seeking more exact design data. *However, vast increases in airplane, missile, and pilotless aircraft performance and new uses of automatic control are making successful and economic design possible only through a detailed knowledge of dynamic behavior.*

An examination of current missile development and that slated for the forseeable future serves to emphasize the wide range of specialized problems that require both a knowledge of aerodynamic response characteristics and a know-how in their application, which have hardly been a part of the traditional aerodynamicist's background. For example, the ability to design guidance equipment for beam- and radar-controlled missiles presupposes sufficient information to perform a closed loop analysis in which the component response characteristics are synthesized for the overall system behavior. This information, while readily available for the electronic elements of a controlling system, has yet to be provided satisfactorily for the aerodynamic ones. In these applications the aerodynamicist is being forced to face such fundamental issues as the time constants relating flight path and attitude changes to control displacement and the result of application of various control forcing functions. He is being required to gear-up, from his essentially static aerodynamics, to such concepts as feedback and complex transfer functions that are used in dynamic electrical and servomechanism design.

In the man-carrying aircraft field the use of dynamic analysis is equally urgent. As automatic control daily finds new uses, its designers stress the need for more aerodynamic response information. Improvement in the rough-air riding qualities and safety of transport aircraft when flown on automatic control will certainly be demanded in the near future, along with advances in push-button automatic flight control methods. Radar bombing and tracking devices must be considered inherently with the dynamics and automatic control of the aircraft in which they are installed. A corollary to the design of aircraft whose performance approaches or extends into the trans- and supersonic régimes is that many features of the external configuration (such as sweepback, etc.) are fixed by the performance requirements themselves, thus enhancing the difficulties of the stability and control designer and necessitating more precise methods of analysis. Formidable dynamic problems obviously face the designer in the transonic region and at the stall. Finally, in the case of the large or unconventional aircraft, the prototype investment may become so huge that, in the light of increasingly stringent flying qualities specifications, it will be no longer economical (or acceptable to the operating agencies) to by-pass a complete dynamic design study—a trend that is already reflected in the incorporation of dynamic requirements in military and civilian flying characteristics specifications.

In short, the manufacturers of either military flying devices or commercial aircraft will, to an increasing degree, be called upon to predict and demonstrate quantitatively the dynamic behavior of their products. On the one hand, it is an inescapable fact that the aircraft is a dynamic machine whose more precise examination is an outgrowth of its widening utility; on the other, it is equally evident that the aerodynamic designer is still not equipped to solve dynamic problems with the desired degree of facility. It is in relation to these facts that the necessity and opportunity for new approaches in dynamic research lie.

Initiation of a Research Program

Two and one-half years ago, from observation of these trends, a project was initiated by the author at the Cornell Aeronautical Laboratory (then the Curtiss-Wright Research Laboratory) as an entrée to a broad research in the field of dynamic stability and control and its application to future flying devices. The overall research, comprising several projects at the present time, and a major endeavor of the Laboratory, has the general objective of furthering the use of dynamic analysis by the aerodynamic designer through a simplification of approach and by the provision of full-scale numerical data. Because of the limited familiarity that most aeronautical engineers have with dynamic stability and control, the project seemed particularly suited to the long-range objective of the Laboratory, which is one of combining research and education. Up to the present time, the research has been entirely performed in the Laboratory's Flight Research Department, which has been, in many respects, organizationally tailored to the unique requirements of dynamic research in flight.

The program has enjoyed, in addition to the early support of Curtiss-Wright, the primary sponsorship of the Aircraft Laboratory, Engineering Division, Air Materiel Command, which has contracted for a major block of the experimental and theoretical work now under way. Currently, a portion of the research (including an application of the flight oscillation technique to a particular design problem) is being performed for the Instrumentation Laboratory at Massachusetts Institute of Technology, while the early exploratory phase of the program was done on a mutually cooperative basis with the Flight Research Branch, Flight Test Division, Air Materiel Command, and the Minneapolis-Honeywell Regulator Company. The Sperry Gyroscope Company, Inc., has extended informal cooperation during a current phase of the program which utilizes their A-12 type of automatic pilot.

It is the purpose of the present report to deal with the general progress of this research, viewed as a whole rather than as individual projects. Being in the nature of a survey, no attempt is made to present more than typical samples of the large amount of flight data that have been collected. Theoretical developments that have progressed in parallel with the flight program are summarized only with respect to the composite picture,

since it is hoped that detailed presentations will be made by their respective authors at an early date. This report is published with the full cognizance of, and appreciation for, the interest of the various sponsoring agencies.

Ideas Underlying the Approach

In initiating a dynamic research program it was, of course, appreciated that a great deal had already been accomplished in the field and that many different avenues of possibility were competently being explored. To avoid an overlapping of effort and ensure an orientation of objectives, a survey was made of previous dynamic research.* An outline of the proposed program[1] was then circulated to interested automatic pilot and aircraft manufacturers, the National Advisory Committee for Aeronautics, and others for their suggestions.

In brief, it was apparent that aircraft dynamic stability and control considerations had for years (and almost without exception) revolved about the so-called classical equations of motion.[2] In these equations, which rigorously hold only for small deviations from steady symmetrical flight, the airplane was taken as a rigid body acted upon by aerodynamic, inertia, and gravitational forces and moments. The unbalanced aerodynamic forces and moments were expressed as the sum of several elements, each of which varied linearly with an increment in a respective linear or angular velocity component, these force and moment curve slopes being referred to as the "stability derivatives." The total forces and moments were largely assumed independent of acceleration components.

In considering further research it was immediately apparent that additional evidence, preferably of a full-scale experimental nature, was needed to substantiate the assumed variation of the aerodynamic forces and moments on modern aircraft. It was later realized that the constancy or the unique existence of the simple velocity derivative was questionable in the more rapid modes of motion, which meant that the actual conditions under which the various types of derivatives would adequately describe the motion were largely unexplored. Furthermore, few measurements had been made in flight to substantiate even the simple individual velocity derivatives as determined from the wind tunnel, largely for the reason that no generally accepted flight-test method had been developed (although a considerable amount of closely allied data, principally static, were obtainable from routine flying qualities tests). Such substantiation, it was felt, might go far toward suggesting new wind-tunnel techniques or the clarification of design criteria for a new type of dynamic wind-tunnel balance; while, on the other hand, a simple scheme for measuring dynamic characteristics in flight would be of outstanding significance for full-scale research at Mach Numbers well beyond the range of dynamic or reliable static wind-tunnel data (it being evident that a complete dynamic determination would include all of the restoring derivatives that correspond to the static stabilities in common use).

One was also led to inquire, in the light of experience in other branches of science (notably electrical, where semimathematical or graphical methods are in use for handling dynamic phenomena), whether there was not some simpler means of expressing the dynamics of the aircraft—at least for certain problems—which would stress the physical aspects, avoid the cumbersome mathematics attendant on the use of the equations with control moments present, and, in general, minimize the limitations of the classical approach. It seemed desirable to tap the broad experience of the electrical and electronics engineers, who are the recognized experts in handling dynamic phenomena. Although most aeronautical designers felt in a vague way that an aircraft was a "spring" system, they had little familiarity with its characteristics even for the more important practical modes.

Finally, it was concluded that almost complete aerodynamic information was lacking which would allow the engineer to match automatic control and aircraft characteristics, a subject that could well be the most important phase of any dynamic investigation.

The above considerations led to the decision that the program should be organized as a full-scale research in flight, backed by closely coordinated analytical investigations. Among the advantages of performing the research in flight was that of being able to observe and experience physically the phenomena under study. (In the author's opinion, "on-the-spot" engineering intelligence is a flight research resource too often overlooked and one of incalculable value in the dynamics field where a consolidation of theory and practice is so important). *It was believed that a thorough understanding of the dynamics of the airplane—which because of its size, flexibility, and speed, is so relatively easy to study and make multiple measurements on—is a first requisite to unraveling the dynamic behavior of the missile.*

It was also decided to use automatic control; first, in the capacity of a flight-test tool (an application originally suggested by Kutzler and Johnston[3]), which would allow the airplane to be flown repeatedly through precise specialized maneuvers; and, second, for the excellent opportunity presented for quantitatively studying its behavior in relation to the aerodynamics of the airplane.

With automatic control present, the most promising method of test appeared to be that of applying known (predetermined) control motions—i.e., control forcing functions—simultaneously recording the airplane response. For some reason this well-known and powerful method of handling dynamic phenomena in general had seen but little aeronautical application.

* Because of its importance to an understanding of the formulation of objectives for the present program, an up-to-date synopsis of this survey is given in the Appendix.

In summary then, it was decided to do the entire job in flight, taking the fullest advantage of whatever automatic control could offer in attacking the problem from the forcing function standpoint. While it was hoped that utilizing electrical engineering experience would suggest ways and means of improving upon the classical theory, attention would first be directed to the determination of the velocity derivatives and the establishment of valid operating limits.

Viewed as an integrated whole, the above approach differed markedly from that of other dynamic research; the additional insurance, it was hoped, that the work would be complementary to allied endeavor.

Summary of Specific Objectives

On the basis of the ideas outlined above, it was possible to state a set of definite long-range objectives. These objectives are as follows:

(1) To develop automatic control and instrumentation methods and techniques for measuring airplane dynamic stability and control characteristics in flight.

(2) To perform comparative tests with existing types of automatic pilots to ascertain characteristics most suitable for their use in item (1); to suggest and test modifications enhancing their use in that respect.

(3) To amass a fund of numerical forcing function response data from flight over a wide range of control application rates and for various aircraft and operating conditions.

(4) To study the relationship between flight-test response information and the classical derivatives; to compare the response information, properly reduced, to wind-tunnel and theoretical information to further confirm or establish limits of its applicability.

(5) To suggest new and simpler theoretical approaches, or wind-tunnel techniques, falling in line with the flight experience and easily usable in practical design.

(6) To study and demonstrate the feasibility of using the experimental response methods in high Mach Number man-carrying airplane applications and in missile and pilotless aircraft design.

(7) To enhance the measurement and application of airplane dynamic stability and control information for use in the design of automatic control, servomechanisms, and power boost.

(8) To provide dynamic information in a form suitable for simulator design or for checking the performance of simulators.

(9) To adapt the dynamic approaches used in other branches of science to the needs and problems of the aircraft designer.

(10) To study and research experimentally upon the application of automatic control for unstable and unconventional aircraft, stall control, ride control, and other specialized flight problems.

(11) To study experimentally, if such appears practical, the combined longitudinal and lateral motion.

(12) To simplify such new automatic control methods as are developed to the point where they may be incorporated into routine flight testing (as opposed to their pure research use); to devise and perfect flight methods for measuring dynamic information which may be applied independently of automatic control.

Many secondary objectives outside the scope of this report were also established.

Organization and Remarks

During the last $2^1/_2$ years, the effort expended on this research has grown to about 60 per cent of that of the Flight Research Department, an organization of 70 people. In addition to the flight crew, man power is available for project, instrumentation, mechanical, photographic, and theoretical and analysis engineering functions. The Flight Research Department is fortunate in possessing all of those organizational elements necessary to a research program in flight and an associated ground analytical investigation in a closely integrated, singly directed organization. Of even more consequence to the present research is the fact that in one relatively small group resides specialized experience in aerodynamics, electronics, and applied mathematics, as well as in flight testing. More than 250 hours of flying have been devoted to the dynamic stability and control work since its inception, the greatest portion of this flying being on service airplanes of the medium bomber category.

Among the problems faced in a research of this nature are those associated with a stacking of the well-known (but often minimized) intricacies of multiple-channel dynamic recording equipment with those of adapting automatic control to new uses. To these there must be added the usual and numerous routine problems of carefully controlled flight testing. Obviously, great care must be taken in the measurements so that the dynamics of the airplane (if under test) are isolated from the dynamics of the automatic control and/or the recording equipment. Brief discussions of some of the interesting instrumentation problems encountered are included in later sections of this report.

The research as a whole has attained many of the goals initially set and has thoroughly established the longitudinal oscillation forcing function technique for obtaining dynamic stability and control data. It is now progressing upon an investigation of the step function approach, the application of the longitudinal technique to particular problems, and the oscillation technique to the lateral and combined lateral and longitudinal motions. While the future direction of the work is discussed in a later section of this report, it may be mentioned here that the program bids well to be a continuing endeavor of increasing magnitude.

Part II—Longitudinal Oscillation Method

Introduction

In accordance with the general method of approach described in Part I, a series of tests were initiated (about 2 years ago) for the determination of the dynamic longitudinal response characteristics of an airplane. The results of these tests were reported[4] at the Fourteenth Annual I.A.S. Meeting but were never prepared for publication. Acknowledgment is made to the coauthors for the use of a part of this material in the present report.

The tests were performed on a North American B-25J airplane equipped with a Minneapolis-Honeywell C-1 type automatic pilot, and the longitudinal stability and control characteristics were obtained from the response of the airplane, as measured by variation in normal acceleration and pitch angle when the elevator was moved with a continuous sinusoidal motion. It will be immediately recognized that the test method is merely one of applying a sinusoidal elevator control forcing function and determining the airplane response by a measurement of the most convenient variables of the motion. The automatic pilot, suitably modified, proved to be a logical device for applying this specialized control motion, which, it should be emphasized, is superimposed upon the basic function of the automatic pilot, thus enabling the maintenance of a desired mean flight path. This superposition has been shown to in no wise affect the results of the tests (which are based on measured elevator angle) if the air conditions are reasonably smooth.

The application of a sinusoidal elevator control forcing function had previously been explored by at least one automatic pilot manufacturer (Sperry), though the present tests probably represent the first time that the technique has been used in the specific determination of full-scale aerodynamic stability and control data. The decision to use the sinusoidal forcing function, in preference to some other type, was based on the desirability of obtaining (during the initial tests, at least) steady-state information. The advantages of working with the longitudinal motion were emphasized on discussion with Koppen, who predicted the relatively low normal acceleration that was obtained for any reasonable elevator motion.

The potentialities of the sinusoidal forcing function need hardly be mentioned. Through its use in mechanical and electrical systems, a large fund of experimental information has been collected for rational design. While there was some doubt as to the reduction of this type of information for direct use in airplane design, its utility in automatic control and servomechanism adaptation has never been questioned.

Instrumentation, Procedure, and Tests

For the purpose of developing the method it was desirable to obtain a continuous time history of all of the variables, while for routine use more preferable means of obtaining the pertinent data may be evolved. For the present tests continuous recordings of the following items were obtained on an oscillograph: (1) elevator angle; (2) elevator hinge moment; (3) pitch angle; (4) normal acceleration at nose, center of gravity, and tail; (5) angle of attack; [(6) bending stresses in horizontal tail spars].

The large gravitationally controlled flight gyro of the automatic pilot itself was used as the source of the pitch angle signal, while Consolidated reluctance-type accelerometers were used for the normal accelerations. The instrumentation was so installed in the compartment adjacent to, and immediately behind, the pilot's location that the operator could also control the automatic pilot and, without moving, observe the open-type photorecorder, the oscillographic equipment, the pilot's instrument panel, and, in addition, see forward and to the side through the copilot's windows. By this strategic location of "intelligence," a much clearer physical picture was obtained than from the mere examination of recorded data. Fig. 1 shows a general view of the instrumentation and other equipment, while Fig. 2a is a typical record from the oscillograph. For these tests no filters for the reduction of hash were installed in the oscillograph channels, although they were later introduced (the final development being shown in Fig. 2b). The elevator motion, obtained by the adaptation of an electronic oscillator control to the automatic pilot, as closely approximated the sinusoidal as could be expected of the basic constant speed servomotor design. Fig. 3 shows a typical time history of the pitch angle, c.g. acceleration, elevator angle, and hinge moment variation.

The test procedure consisted of stabilizing at a given flight condition and establishing an oscillation of small amplitude (approximately $\pm 1^1/_2°$ elevator angle) at a given frequency. When steady-state conditions were attained, simultaneous oscillographic and photo-observer records were made for at least five complete oscillation cycles. This procedure was repeated for the entire band of frequencies, ranging in period from 1 to 7 sec. and giving approximately 25 points on the response curve for a given flight condition. Tests were performed in level flight, climb, and descent, but for the present purposes only the level flight runs at equivalent air speeds* of from 135 to 215 m.p.h. will be discussed. (Excellent data have, however, been obtained in climbs and descents.)

It was found that the airplane oscillated steadily with no sensible change in air speed or longitudinal acceleration and that for any reasonable value of elevator amplitude ($< \pm 3°$) the normal acceleration increment at the center of gravity was less than $\pm 0.6g$. Because of the combined rotation and vertical translation, the apparent center of rotation of the airplane, as seen in

* Equivalent air speed is defined as the true air speed multiplied by the square root of the density ratio.

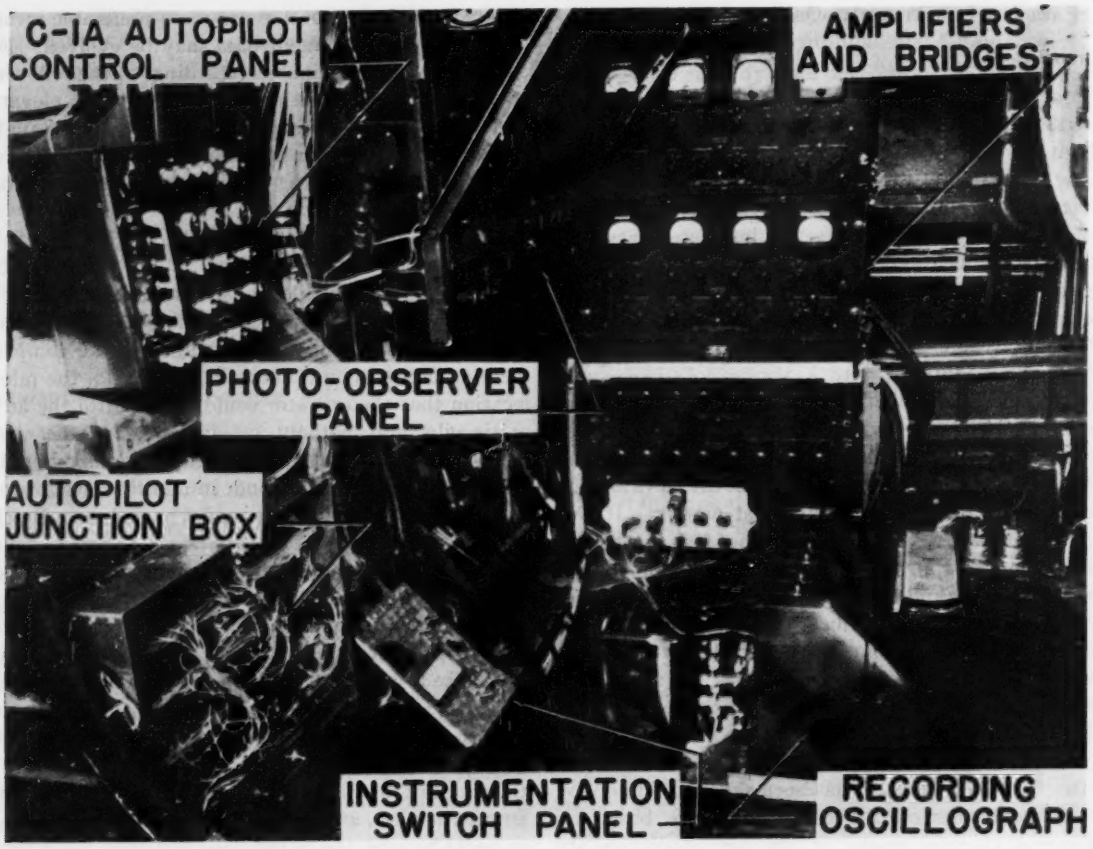

FIG. 1. General view of instrumentation, B-25J airplane.

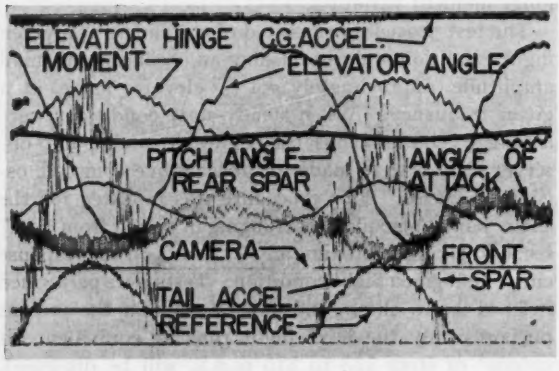

FIG. 2a. Typical oscillograph records from flight. Early records with no electrical filter.

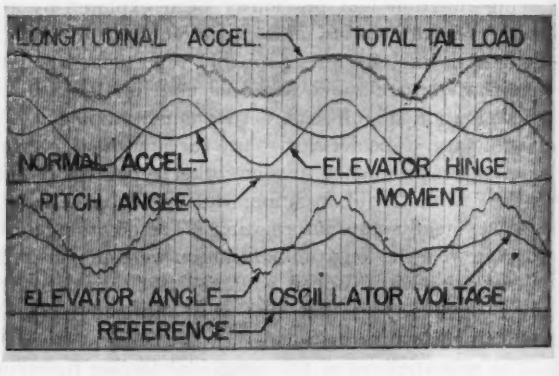

FIG. 2b. Typical oscillograph records from flight. Recent records with electrical filters.

reference to the horizon, was about 6 ft. ahead of the center of gravity and moved progressively forward with increases in air speed. While these figures are to some extent functions of airplane size, speed, etc., they serve to indicate that the method is physically practical. When the airplane was oscillated for a sufficiently long period, an approach to the phugoid mode was noticeable in that the air speed and altitude no longer remained exactly constant. (Further discussion of this point is presented later.)

Experience with the method has indicated that the instrumentation and procedure are basically satisfactory and that, in general, it is a technique satisfactory throughout the speed range and in any attitude normally experienced in this class of airplane. The test scatter of the pitch angle data was later reduced by

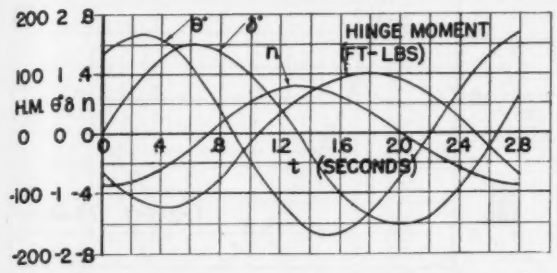

FIG. 3. Experimental time history of elevator angle, pitch angle, resultant normal acceleration of c.g., and elevator hinge moment for B-25J airplane at 175 m.p.h. equivalent air speed $\omega_f = 2.36$ rad. per sec.

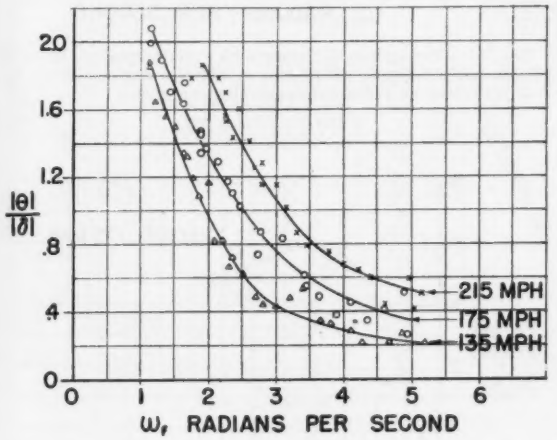

FIG. 4. Experimental variation of pitch angle amplitude with frequency of elevator oscillation for the B-25J airplane.

suitable modification to the instrumentation, improving the readability of the data from the oscillograph record.

Results

Basic experimental results from these tests are shown in Figs. 4 and 5, where the magnitudes of the pitch (or attitude) angle (θ, in degrees), and the resultant normal acceleration at the center of gravity (n, in g units), is in each case divided by the magnitude of the elevator angle (δ, in degrees) and plotted vs. frequency in radians per second. These data are all reduced from time histories (as of Fig. 3) by halving measured double amplitudes averaged over a few cycles of the motion. The scatter of the test points, particularly in the case of the pitch angle, has since been reduced as experience in the instrumentation and data reduction has been acquired. Unfortunately, in these preliminary tests, the scatter did not permit accurate phase-angle measurement for the tests under discussion.

Analysis and Discussion

During the tests it had been observed that, throughout the frequency range covered, no noticeable change occurred in the air-speed indicator reading. Although

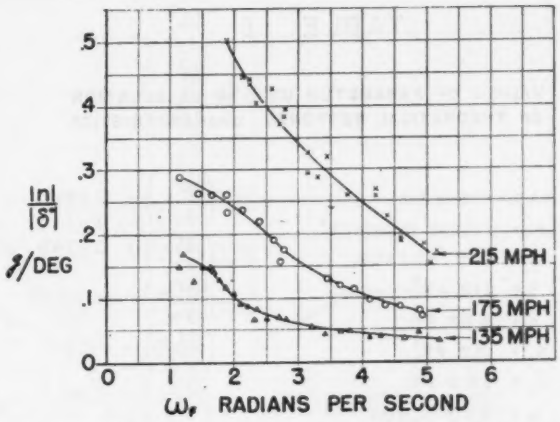

FIG. 5. Experimental variation of resultant normal acceleration of center of gravity with frequency of elevator oscillation for B-25J airplane.

air-speed systems are notorious for lag, it was nevertheless felt that the motion was occurring at substantially constant speed (an assumption that has since been substantiated is discussed further in Part III). It was then possible to make a simplified theoretical comparison with the test data by eliminating the X equation and such terms as would arise from speed variation in the remaining classical longitudinal stability equations. If the dimensional equations* of reference 2 are thus modified, and the elevator forcing function, $\delta M_\delta \times \sin \omega t$, introduced (on the assumption that the forcing function is entirely one of pitching moment), a solution may be obtained by straightforward methods for the amplitudes of the pitch angle and acceleration response; thus,

$$\frac{|\theta|}{\delta} = M_\delta \sqrt{\frac{1 + (Z_w/\omega)^2}{\omega^4 + (C_1^2 - 2C_2)\omega^2 = C_2^2}}$$

$$\frac{|\dot{w}|}{\delta} = M_\delta \frac{U\omega}{\sqrt{\omega^4 + (C_1^2 - 2C_2)\omega^2 + C_2^2}}$$

where

$$C_1 = Z_w + M_q + M_{\dot{w}} U$$
$$C_2 = Z_w M_q - M_w U$$

and

$$|\dot{w} - U\dot{\theta}|/\delta = |n|/\delta$$

U = steady forward velocity
n = resultant normal acceleration as measured by accelerometer at airplane center of gravity
δ = amplitude of elevator angle
θ = angle of longitudinal airplane reference axis to horizontal
w = velocity increment along the Z axis
M_δ, Z_w, M_q = dimensional stability derivatives

* For many comparative computations the use of the dimensional equations has been found convenient.

TABLE I

VALUES OF PARAMETER USED IN CALCULATION OF THEORETICAL RESPONSE CHARACTERISTICS

E.A.S. = 175 M.P.H.
ρ = 0.00165 SLUGS/FT.^{3}
U = 308 FT./SEC.
S = 610 FT.^{2}
c = 9.67 FT.
S_t = 132 FT.^{2}
l_t = 25.9 FT.
m = 840 SLUGS
I_y = 50,000 SLUG FT.^{2}

$\left(\frac{\partial C_L}{\partial \alpha^\circ}\right)$ = 0.0816
$\left(\frac{dC_L}{d\alpha^\circ}\right)$ = 0.0769
$\left(\frac{\partial C_m}{\partial \alpha^\circ}\right)$ = −0.0049
$\left(\frac{\partial C_m}{\partial \delta^\circ}\right)$ = −0.017
$\left(\frac{d\epsilon}{d\alpha}\right)$ = 0.45
η_t = 1.0 (POWER ON)

$$Z_w = -\left(\frac{\partial C_L}{\partial \alpha}\right)\frac{1}{m}\frac{\rho U S}{2} = -0.864 \text{ 1/SEC.}$$

$$M_w = \left(\frac{\partial C_m}{\partial \alpha}\right)\frac{1}{I_y}\frac{\rho U S c}{2} = -0.00842 \text{ 1/FT. SEC.}$$

$$M_{\dot{w}} = -\left(\frac{dC_L}{d\alpha}\right)_t\left(\frac{d\epsilon}{d\alpha}\right)\frac{1}{I_y}\frac{\rho S_t l_t^2 \eta_t}{2} = -0.00292 \text{ 1/SEC.}$$

$$M_q = -\left(\frac{dC_L}{d\alpha}\right)_t\frac{1}{I_y}\frac{\rho U S_t l_t^2 \eta_t}{2} = -1.99 \text{ 1/SEC.}$$

$$Z_q = \frac{1}{l}\frac{I_y}{m}M_q = -4.57 \text{ FT./SEC.}$$

$$M_\delta = \left(\frac{\partial C_m}{\partial \delta}\right)\frac{1}{I_y}\frac{\rho U^2 S c}{2} = -8.98 \text{ 1/SEC.}^2$$

Numerical values of the various parameters calculated from the best wind-tunnel and theoretical data available and for an equivalent air speed of 175 m.p.h. are summarized in Table I, while Figs. 6 and 7 show the agreement that was obtained with the experiment. A number of other comparisons have been made, including that for a complete cycle of oscillation, and all of them show excellent agreement.

While a check had been established between the measured response and conventional theory, a far more significant possibility—that of obtaining the derivatives directly from the flight tests—had been opened. Thus, a method for determining these aerodynamic constants at high Mach Numbers, or under other conditions beyond the scope of wind-tunnel data and theory, would be extremely worth while. If the equation previously given for $|\theta|/\delta$ is divided by that for $|\dot{w}|/\delta$, a simple relationship results from which Z_w, or the slope of the airplane lift curve $dC_L/d\alpha$, may be obtained directly. The other derivatives, including the controlling moment, M_δ, could be extracted from the amplitude and phase relationships, provided the test data were of a sufficient accuracy. Such a process had been contemplated for the present tests, but the instrumentation inaccuracies

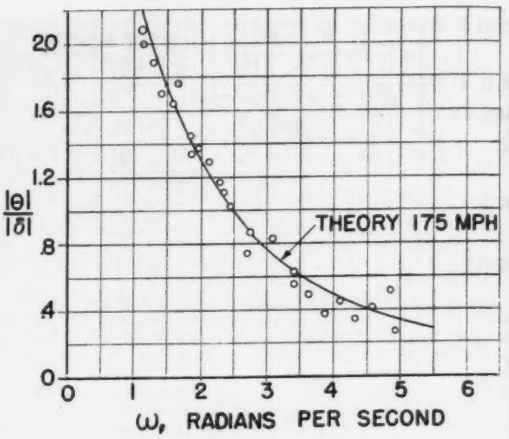

FIG. 6. Comparison of theory with experimental variation of pitch angle amplitude with frequency of elevator oscillation for B-25J airplane at 175 m.p.h. equivalent air speed.

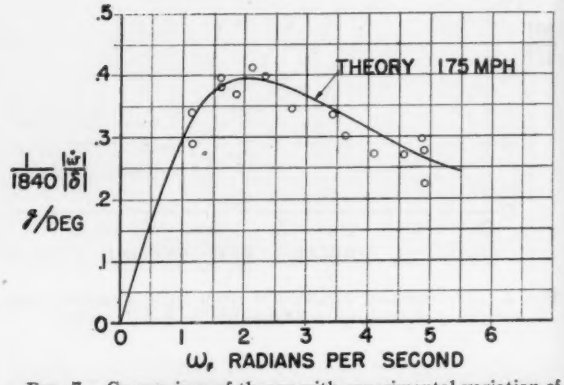

FIG. 7. Comparison of theory with experimental variation of normal translational acceleration of the center of gravity with frequency of elevator oscillation for the B-25J airplane at 175 m.p.h. equivalent air speed.

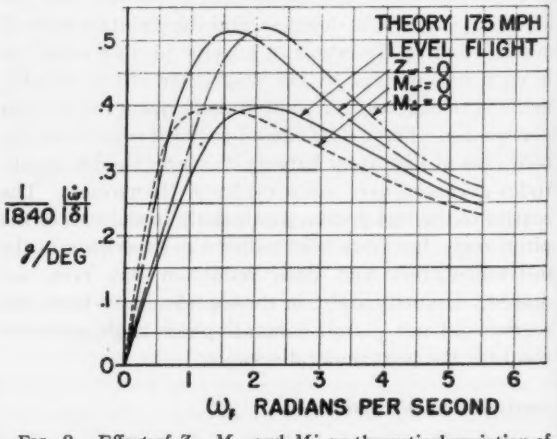

FIG. 8. Effect of Z_w, M_w, and $M_{\dot{w}}$ on theoretical variation of normal translational acceleration of the center of gravity with frequency of elevator oscillation for the B-25J airplane at 175 m.p.h. equivalent air speed.

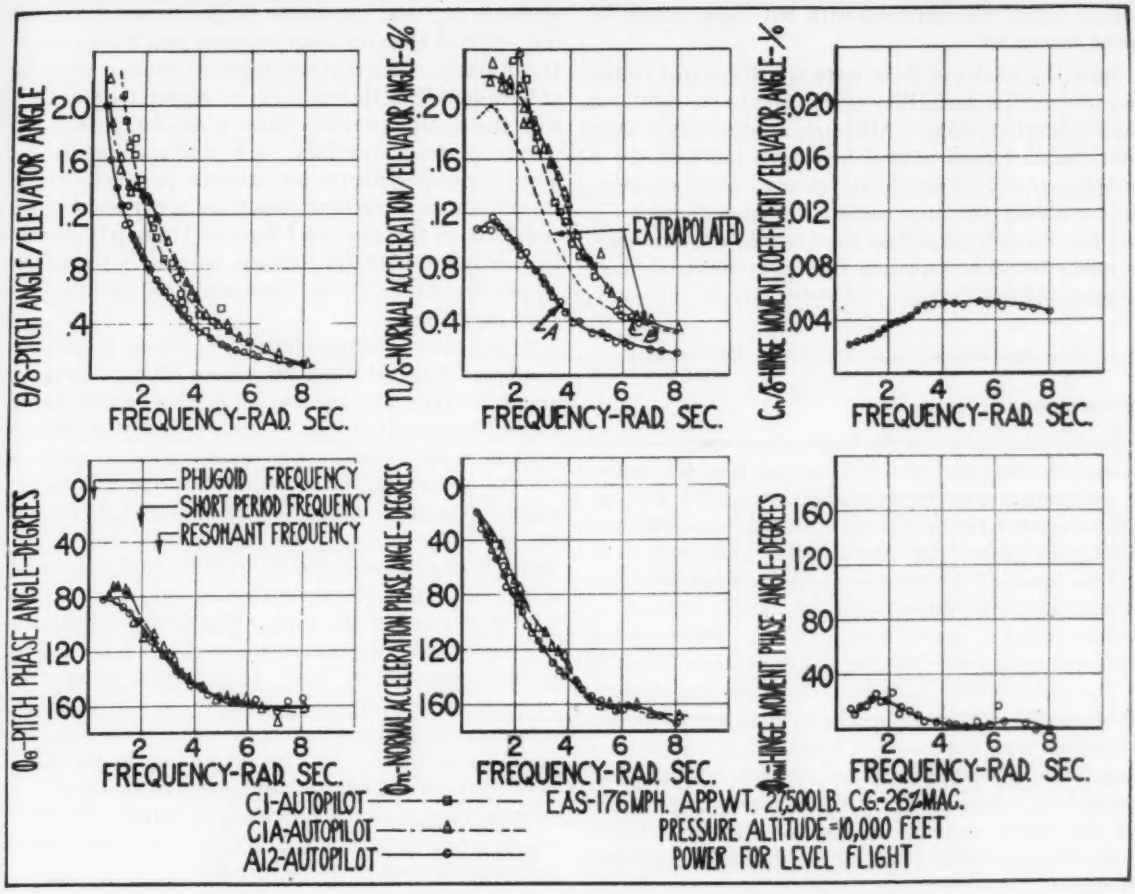

FIG. 9. Basic response data with various automatic pilots, B-25J airplane.

(referred to previously) and resultant lack of phase data made this prohibitive.

In lieu of the above possibility and to establish accuracy limits for future work, response curves were calculated in which each derivative was separately set equal to zero. The resultant displacement of the response curve gives a practical idea of how closely it would have to be established experimentally if the derivatives were to be extracted. Fig. 8 shows the result of such an investigation on the normal acceleration response for the derivatives Z_w, M_w, and $M_{\dot{w}}$.

The conclusion drawn from studies of this kind was that a high but not impractical order of accuracy would be required in experimental work directed at this objective.

Concluding Remarks

While it is desirable to reduce the response measurements to basic aerodynamic stability and control parameters, direct uses suggest themselves, such as the comparison of aircraft through nondimensional response parameters. Many flying qualities, now measured and specified only for quasi-static conditions, stand to be generalized by determination over the complete range of control operating frequencies. In fact, sinusoidal testing may be a logical way of expressing those items of behavior which vary with pilot rate of control application. (From the present tests stick force/g may be plotted vs. frequency and extrapolated to zero frequency to check the stick force/g as normally measured in wind-up turns and steady pull-ups.) It is possible that the concept of frequency may lead to a more adequate definition and understanding of maneuverability. For automatic control work, simple functions of the measured response appear most desirable.

For conventional aircraft the longitudinal motions of primary operational import are those that occur at sensibly constant speed. At the lowest frequency of the present tests the speed change from trim was less than 2 per cent. A fixed control airplane in symmetrical flight definitely behaves as a two-degree-of-freedom system, whose motions cannot be approximated by the assumption of one-degree. This was convincingly shown by comparing theoretical curves based on both assumptions with the experimental data. In future tests, the fre-

quency range was increased with but slight change in speed variations.

Since the results of these tests are all referred to the elevator motion itself, they correspond to conventional fixed control stability. Although measurements were later made of total normal horizontal tail load (by a complex strain gage setup) and elevator hinge moment, thus enabling the approximate determination of C_{h_α} and C_{h_δ}, the reduction from fixed to free was laborious. A more desirable approach to the problem of hinge moment response has since been discovered.

Part III—Subsequent Longitudinal Developments

Introduction

In the last $1^1/_2$ years the longitudinal work outlined in the preceding part of this report has been continued. In accordance with the original objectives (2 and 3) oscillation tests have been completed with two additional automatic pilots—the Minneapolis-Honeywell C-1A and the Sperry A-12—and a considerable fund of longitudinal response information has been amassed at various center of gravity locations, etc. This work has all been accomplished on the B-25J airplane, although at the present time a Douglas A-26 airplane is completely instrumented and will be used, in part, to obtain additional and comparative information. Simultaneously with the testing program, methods have been developed for reducing the response data in terms usable for automatic control and servomechanism design, with continued effort on its utility in the straight aerodynamic design of aircraft (in line with objectives 4, 5, 6, and 7).

It will be appreciated that, as an overall endeavor of this sort grows in magnitude and as new minds are brought to bear on the problems, the number of ideas and possible directions for investigation fan out rapidly. In the present part of the report an attempt will be made to organize and discuss only those ramifications that currently promise to fit into the permanent picture. These developments were participated in by a number of people, and, where practical, direct acknowledgment is given.

No attempt will be made to evaluate the relative merits, in the present application, of the various automatic pilots. Such comparative curves as have been chosen for this report will not enable any conclusions, in this respect, to be drawn.

Experimental Oscillation Data

A set of basic longitudinal response curves, showing test points, is presented in Figs. 9a–9f. Insofar as the data were available, points are shown for the different automatic pilots. While in the earlier tests the response was measured only in terms of the parameters $|\theta|/\delta$ and $|n|/\delta$, improvements and additions in instrumentation now enable the inclusion of the elevator hinge moment response C_{h_δ}, and the phase angles between θ and δ (i.e., ϕ_θ) and elevator hinge moment and δ (i.e., ϕ_{HM_E}). It will also be noted that the frequency range is extended at both ends from that used in the original tests; otherwise the instrumentation setup is similar, though improved in detail (see Figs. 2), to that used previously. The information for the six response parameters is, of course, obtained simultaneously for a particular flight condition, in this case level flight at 175 m.p.h. equivalent air speed, 10,000 ft. altitude, with a c.g. location of 26 per cent M.A.C.—the "standard" condition for the B-25 program.

In the case of the phase angles, ϕ_θ and ϕ_n, good agreement was obtained when two totally different types of automatic pilot were used, and a number of months had elapsed between the tests. This agreement is perhaps even more remarkable when consideration is given to the size of the filter correction which it was necessary to apply to the phase angles. In the case of $|\theta|/\delta$ and $|n|/\delta$ the agreement is not so close as could be desired, particularly in the low-frequency range. In Fig. 9, curve A has been given to illustrate the care that must be used in research of this type. From an examination of the regularity of the points, one might be led to believe in the validity of the data. It was suspected of error, however, before the present comparison was made, and a section of the frequency range was repeated with results shown at B. The peculiar hump in the C-1A, $|\theta|/\delta$ response, which initially gave rise to many conjectures, was traced to an oscillator detail. The hinge moment phase angle, ϕ_{HM_E}, shows a small "lead," which is the result of the C_{h_α} component.

A study of the experimental data in the light of theory has been of the greatest assistance in establishing its validity and, hence, in assuring a satisfactory test technique. It is, of course, generally possible to establish an approximation to the parameters for static ($\omega = 0$) and for infinite frequency conditions.

A second set of longitudinal response information is presented in Figs. 10a–10f, where data are shown for several air speeds at constant power. The data for 135 m.p.h. equivalent air speed were therefore obtained in climb, while the data for 200 m.p.h. equivalent air speed were obtained in dive. These curves were well established by numerous points as for the preceding set.

The two sets of data have been largely shown to illustrate the amount of information oscillation runs yield and the manner in which the basic data are conveniently plotted and faired. Plotting the parameters directly against frequency is not, however, always useful in judging the effect of an independent variable. In Fig. 10b the effect of speed on $|n|/\delta$ is indicated satisfactorily, but in Fig. 10e the curves are not so simply displaced. The use of a frequency ratio as the abscissa may be desirable for some cases, such as for data at various center of gravity locations.

To illustrate further the wide range of frequencies over which the present tests have been run, the fre-

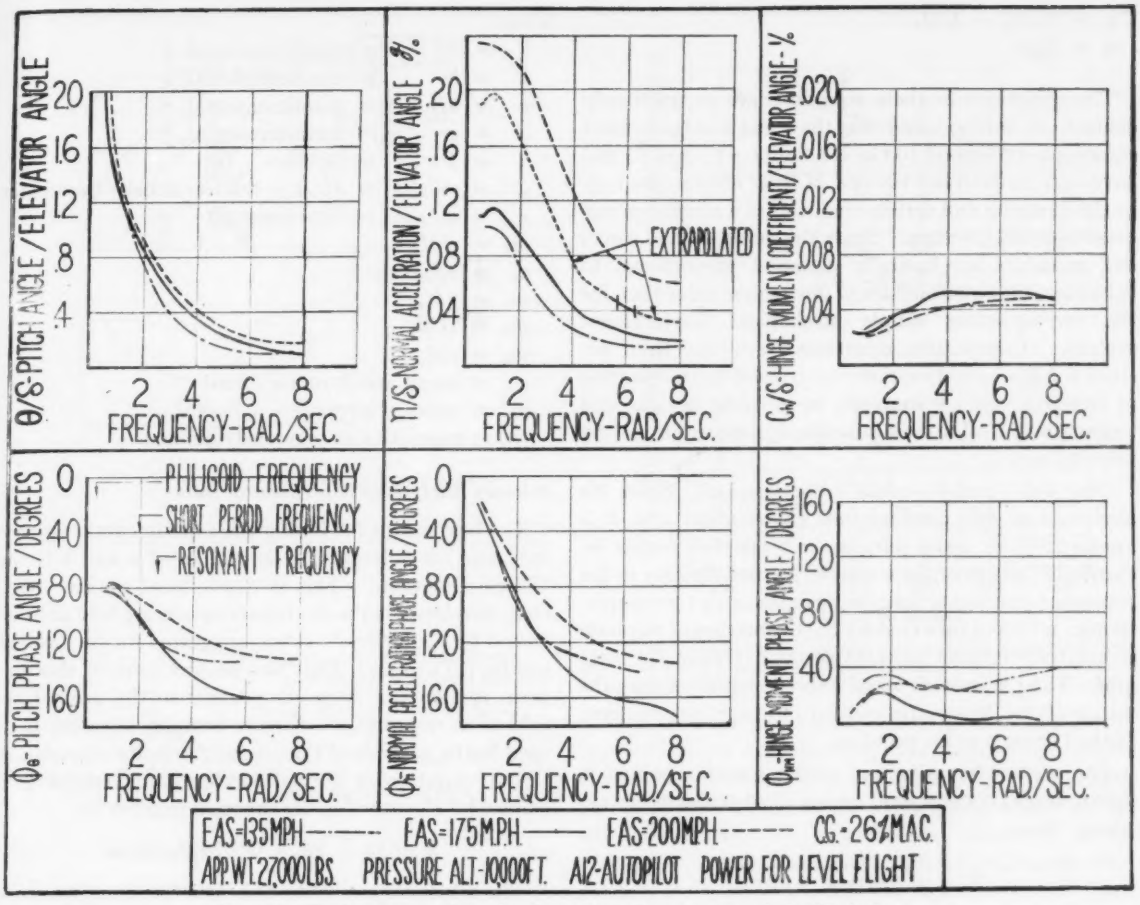

FIG. 10. Basic response data showing effect of varying speed, B-25J airplane.

quencies for the fixed control short and long (phugoid) natural oscillations of the airplane have been indicated on Figs. 9 and 10.

An item of practical importance in obtaining data of this type is the reduction of hash in the oscillograph galvanometer circuits. Electrical filters are used to reduce this hash and have response characteristics of their own which must be known to apply properly corrections to the data. *This is illustrative of the necessity of understanding the dynamics of all the systems involved in research of this type.* It appears to be possible to design filters whose characteristics in amplitude and phase, in the test range of frequencies, are such that corrections to the data may be omitted. However, if this is not possible, then calibrations and corrections must be made. In the case of transient responses, the correct filter design is even more paramount, since the corrections are difficult to apply if the filter responses are large and nonlinear.

Theoretical Developments

Once the experimental longitudinal response data have been adequately established, the matter of its interpretation and use arises. Faced with this problem in the case of a specific application, a first reaction was to reduce the data, if possible, to the derivatives, whose values could then be inserted into the classical equations. In the application concerned—a feedback system—it was, however, obviously desirable to express the response as simply as possible, and, physically, it was felt that for the constant speed motion a simpler means must be available.

E. V. Laitone[5] has suggested an approach that has proved extremely useful in this connection. The solutions for θ and n (as given under Analysis and Discussion, Part II) for motion at constant speed indicated that each varies directly with the applied elevator moment δM_δ. This being the case, it is possible to combine the two equations of motion, rather than resort to the usual simultaneous solution. The final relationships for the pitching angular velocity, q, and the resultant normal acceleration, n, are

$$\ddot{q} + b\dot{q} + kq = M_\delta(\dot{\delta} - Z_w\delta)$$
$$\ddot{n} + b\dot{n} + kn = UM_\delta Z_w\dot{\delta}$$

where

$$b = -(Z_w + M_\dot{\theta} + UM_w)$$

$k = Z_w M_q - UM_w$
$n = Z_w w$

The quantities in these equations are as previously defined. It will be noted that the b and k of the present equations correspond to the constants $-C_1$ and C_2 and have been resymboled because of their obvious analogy to the damping and spring constants of a simple spring-mass-resistance system. Since the aerodynamic forces and moments are basically assumed proportional to velocities, the equations are of one higher order than for the corresponding simple mechanical displacement system. (Comparable equations have also been derived for the lateral motion constrained to two degrees of freedom—as, for example, by sensing devices and automatic control to maintain sideslip, rolling, or yawing velocity zero.)

Once the b and k—which aerodynamically define the motion—have been obtained from experimental data, it is unnecessary for many purposes to accomplish further reduction. They may, for example, be used directly in the response equations or for computing the step function response, which in turn enables the evaluation of response to arbitrary control motions through Duhamel's integral. The parameters b and k should largely satisfy the wants of the automatic control designer for the fixed control aspects of his problem.

The effective damping and spring constants, b and k, define the longitudinal fixed control short period of the airplane; thus,

$$P_{\text{sec.}} = 2\pi/\sqrt{k - (b/2)^2}$$

For the B-25J, center of gravity at 26 per cent, the value of b is 3.7 and $k = 4.3$. Calculations made using the B-25J as a standard, and varying design parameters such as tail length, horizontal tail area, etc., indicate that the dimensional b and k may vary between 2 and 7 for this class airplane. A 20 per cent increase in tail length for a given "tail volume" would critically damp the motion that, for most satisfactory airplanes, is probably never far from being critically damped even for elevator free. The spring constant k is seen to be not only a function of the static pitching moment slope but also of the damping in pitch and the slope of the lift curve. If the center of gravity is moved progressively aft, the value of k will become zero at approximately 39 per cent for the B-25J, which is aft of the fixed control neutral point and in the vicinity of the sea level "maneuver" point.

For comparison purposes the nondimensional form of the equations may be more useful. These are given below:

$$(d^2 + \bar{b}d + \bar{k})\bar{q} = \delta m_\delta (d - z_w)$$
$$(d^2 + \bar{b}d + \bar{k})\bar{n} = \delta m_\delta z_w$$
$$\bar{b} = -(z_w + m_q + \mu m_{\dot{w}})$$
$$\bar{k} = z_w m_q - \mu m_w$$

where

$\bar{b}$	$= b\tau$	$=$	nondimensional, b
$\bar{k}$	$= k\tau^2$	$=$	nondimensional, k
$\bar{n}$	$= n(\tau/U)$	$=$	nondimensional, n
$\bar{q}$	$= q\tau$	$=$	nondimensional, q
d	$= D\tau$	$=$	$(d/dt)\tau$
μ	$= m/\rho Sc$	$=$	$\tau U/c$ = relative density parameter
τ	$= m/\rho SU$	$=$	time constant
m_δ	$= \tau^2 M_\delta$		
z_w	$= \tau Z_w$		
m_q	$= \tau M_q$		
m_w	$= c\tau M_w$		
$m_{\dot{w}}$	$= cM_{\dot{w}}$		
c	$=$ mean aerodynamic chord		
m	$=$ mass of airplane		

Other quantities as previously defined.

Primary Analysis of Experimental Data

The best means for fairing the experimental response data and for determining the values of b and k is the "Circle Diagram." This is a graphical scheme originally developed in the electrical engineering field and discussed by Kennelley.[6] It was suggested for the present use by I. G. Ross. Only the simpler form of this diagram is discussed here—in relation to the normal acceleration response, n. A more complex form has been used in the analysis of the pitching velocity response, q.

The variation of normal acceleration response to a sinusoidal elevator forcing function is given by

$$\ddot{n} + b\dot{n} + kn = |\delta| UM_\delta Z_w \sin \omega t$$

The solution for n, obtained from this equation, may be differentiated and simplified to give:

$$\frac{\dot{n}}{|\delta|} = \frac{UM_\delta Z_w}{\sqrt{[(k/\omega) - \omega]^2 + b^2}} \times \sin\left[\omega t + \tan^{-1}\frac{(k/\omega) - \omega}{b}\right]$$

where $\tan^{-1}\{[(k/\omega) - \omega]/b\} = \phi_{\dot{n}}$ = phase angle between $\dot{n}$ and δ.

The maximum amplitude of $\dot{n}/|\delta|$ for a given frequency may then be shown to be

$$|\dot{n}|/|\delta| = (UM_\delta Z_w/b)\cos\phi_{\dot{n}}$$

This is the polar equation of a circle passing through the origin and having its center on the polar axis, where $|\dot{n}|/|\delta|$ is the length of the polar vector, $\phi_{\dot{n}}$ is the polar angle, and $UM_\delta Z_w/b$ is the diameter. It can be shown that for $\phi_{\dot{n}} = 0$ (when the vector radius is the diameter of the circle), the frequency, ω, is equal to the undamped natural frequency of the system, ω_0 and $k = \omega_0^2$.

The value of b can be found from

$$b = (k/\omega') - \omega'$$

where ω' is the circular frequency at which $\phi_{\dot{n}} = +\pi/4$. These relationships are illustrated in Fig. 11.

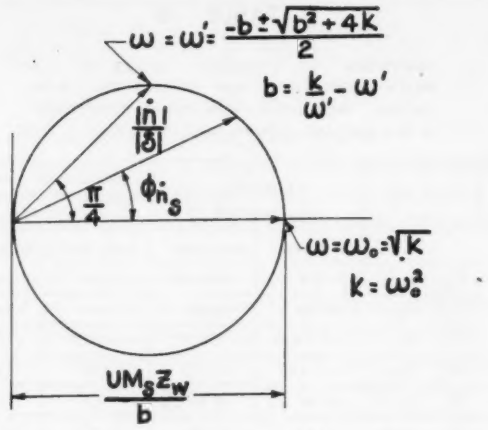

FIG. 11. General properties of the circle diagram.

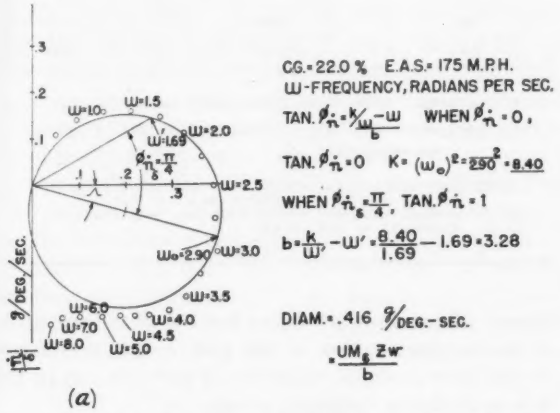
CG. = 22.0 % E.A.S. = 175 M.P.H.
ω-FREQUENCY, RADIANS PER SEC.
TAN. $\phi_{\dot{n}} = \frac{k}{\omega} - \omega$ WHEN $\phi_{\dot{n}} = 0$,
TAN. $\phi_{\dot{n}} = 0$ $K = (\omega_o)^2 = \overline{2.90}^2 = 8.40$
WHEN $\phi_{\dot{n}_\delta} = \frac{\pi}{4}$, TAN. $\phi_{\dot{n}} = 1$
$b = \frac{k}{\omega'} - \omega' = \frac{8.40}{1.69} - 1.69 = 3.28$
DIAM. = .416 g/DEG.-SEC.
$= \frac{UM_\delta Z_w}{b}$
g/DEG/SEC.
(a)

FIG. 12a. Circle diagram presentation of experimental data: $|\dot{n}|/|\delta|$ and $\phi_{\dot{n}_\delta}$ as a function of ω.

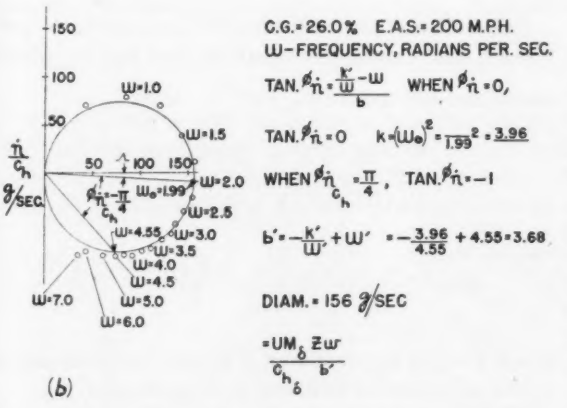
C.G. = 26.0% E.A.S. = 200 M.P.H.
ω-FREQUENCY, RADIANS PER SEC.
TAN. $\phi_{\dot{n}} = \frac{k'}{\omega} - \omega$ WHEN $\phi_{\dot{n}} = 0$,
TAN. $\phi_{\dot{n}} = 0$ $k = (\omega_o)^2 = \overline{1.99}^2 = 3.96$
WHEN $\phi_{\frac{\dot{n}}{C_h}} = \frac{\pi}{4}$, TAN. $\phi_{\dot{n}} = -1$
$b' = -\frac{k'}{\omega'} + \omega' = -\frac{3.96}{4.55} + 4.55 = 3.68$
DIAM. = 156 g/SEC
$= \frac{UM_\delta Z_w}{C_{h_\delta} b'}$
g/SEC.
(b)

FIG. 12b. Circle diagram presentation of experimental data: $|\dot{n}|/|C_h|$ and $\phi_{\dot{n}C_h}$ as a function of ω.

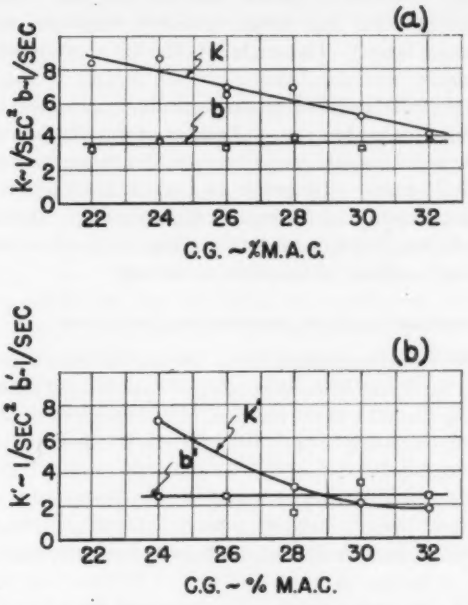

FIG. 13. Experimental variation of stability parameters with center of gravity position, B-25J airplane, 175 m.p.h. equivalent air speed, 10,000-ft. pressure altitude. (a) k and b. (b) k' and b'.

In actual use the values of $|n|/|\delta|$ are obtained directly from the experimental data. The theory shows that these can be converted to values of $|\dot{n}|/|\delta|$ by multiplying by the frequency, ω. The phase angles $\phi_{\dot{n}}$ are determined from the experimental values of ϕ_n by adding 90°. Pairs of values of $|\dot{n}|/|\delta|$ and $\phi_{\dot{n}}$ are then plotted, and a circle is faired through the points. Once the circle has been established (and it is, of course, of great assistance to know that the experimental data should fair in this manner), the values of b and k are determined from the simple relationships given above.

The experimental data when treated in this manner show two interesting features. First, the diameter of the circle that best fits the data does not fall on the polar axis, and the angular difference is in the nature of an aerodynamic lead or lag between the elevator angle and forcing function moment. The assumptions involved in plotting and reducing the data about this displaced axis are being further examined. Second, for very low and very high values of the frequency, the data departs from the circle. In the low-frequency range the departure may be partly associated with the speed change noted in flight; at the high frequencies, where the speed is constant, it is probably indicative of the fact that the simple velocity derivatives are no longer sufficient to describe the motion. Over the large portion of the frequency range, wherein the data fair adequately, assurance can be had that: (1) b and k exist, (2) that the two-degree-of-freedom equations are in line with the physical facts, and (3) that the classic velocity derivatives are probably also existent.

Fig. 12a is an actual reduction of experimental data by the circle diagram method. (Fig. 12a, which is for

"fixed" control response, should not be confused with Fig. 12b, the hinge moment response, to be discussed later.) The angle Λ is the lag mentioned previously.

Fig. 13a is a plot of experimentally determined b and k values for a wide range of center of gravity locations. Each point on this graph represents a reduction by the circle diagram; the graph as a whole therefore represents a substantial amount of flight testing. It will be noted that b is substantially independent of center of gravity location, as would be expected.

Determination of the Aerodynamic Derivatives

The results obtained from the use of the circle diagram for b and k are indicative of the plausibility of reducing the data to the familiar aerodynamic derivatives. In the frequency range, wherein the circle diagram fits the experimental data, such a reduction would logically utilize faired values of b and k (vs. center of gravity location), which could then be thought of as intermediate parameters whose initial fairing increases the accuracy of further reduction.

Subsequent to the establishment of the parameters b and k, mathematical analysis yielded similar constants b' and k', which defined the response of the airplane to applied elevator hinge moment. These parameters are discussed further under "Miscellaneous Developments" but are mentioned here because they are not only useful in isolating the aerodynamic derivatives but in enabling the determination of other useful aerodynamic parameters.

The procedure used in the reduction will now be described. (For convenience in handling the data, all results are in terms of the dimensional quantities.)

It has been previously noted (Part II) that a simple relationship for Z_w may be found by dividing the θ and $\dot{w}$ responses. In terms of the more convenient parameters n and q, this relationship is:

$$|n|/|q| = U/\sqrt{1 + (\omega/Z_w)^2}$$

Fig. 14 shows Z_w as a function of frequency, from experimental data at a number of center of gravity locations. These data are again for the B-25J airplane. From an examination of this plot, -1.25 was taken as the most probable value for 26 per cent c.g. in the low to moderate frequency range.

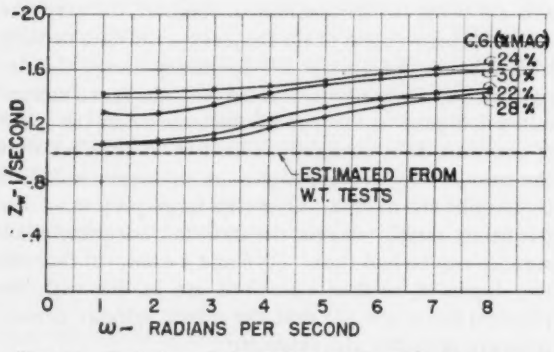

FIG. 14. Variation of the derivative Z_w with frequency of oscillation, B-25J airplane, 175 m.p.h. equivalent air speed, 10,000-ft. pressure altitude.

TABLE II

COMPARISON OF ESTIMATED VALUES OF STABILITY DERIVATIVES AND PARAMETERS WITH VALUES CALCULATED FROM FLIGHT TEST DATA B-25J AIRPLANE, 175 MPH. EAS., 10,000 FOOT ALTITUDE

		ESTIMATED C.G. POSITION		CALCULATED FROM FLIGHT TEST DATA C.G. POSITION	
		26% MAC	28% MAC	26% MAC	28% MAC
$Z_{w'}$	1/SEC.	-1.00 *	-1.00 *	-1.25	-1.08
$M_{w'}$	1/SEC.	-0.0156 *	-0.0125 *	——	——
$M_{\dot{w}'}$	1/FT.	-0.0023	-0.0023	-0.0015	-0.0007
$M_{q'}$	1/SEC.	-1.55	-1.55	-1.80	-2.20
M_{δ}	1/SEC.^{2}	-8.64 * **	-8.64 * **	-6.90	-8.00
$\frac{d\epsilon}{d\alpha}$	——	0.45	0.45	0.75	0.70
C_{h_δ}	1/DEG.	-0.0039 **	-0.0039 *	-0.0054	-0.0054
$C_{h_{tac}}$	1/DEG.	-0.0035 **	-0.0035 *	-0.0046	-0.0041
k	1/SEC.^{2}	6.21	5.28	6.90	6.10
b	1/SEC.	3.24	3.24	3.50	2.50
k'	1/SEC.^{2}	1.27	0.34	4.80	3.70
b'	1/SEC.	2.27	2.27	2.60	2.60

* OBTAINED FROM WIND TUNNEL TEST DATA.

** OBTAINED FROM STATIC STABILITY FLIGHT TEST MEASUREMENTS.

NOTE: SOME VALUES GIVEN IN THIS TABLE WILL NOT COMPARE EXACTLY WITH THE CORRESPONDING VALUES OF TABLE I. IMPROVED DATA WERE AVAILABLE FOR THE CALCULATIONS SUMMARIZED IN THIS TABLE

The value of M_δ may now be determined from the diameter of the fixed control circle diagram which equals $UM_\delta Z_w/b$. The corresponding diameter of the hinge moment circle diagram (the $|\dot{n}|/C_h$ circle) is equal to $UM_\delta Z_w/C_{h_\delta} b'$ and permits of the isolation of C_{h_δ}.

Subtraction of the mathematical relations for b' and b and k' and k yields two equations that may be solved simultaneously to give C_{h_α} and $\frac{d\epsilon}{d\alpha}$; thus,

$$C_{h_\alpha} = \frac{(b' - b) + (l/U)(k' - k)}{\left(\frac{57.3M_\delta}{C_{h_\delta}}\right)\left(\frac{l}{U}\right)\left(2 - \frac{l}{U}Z_w\right)}$$

$$\frac{d\epsilon}{d\alpha} = -\frac{(k' - k)}{\left(\frac{57.3M_\delta}{C_{h_\delta}}\right)C_{h_\alpha}} + \left(1 - \frac{l}{U}Z_w\right)$$

where l = tail length (center of gravity to aerodynamic center of horizontal tail) and M_δ is per degree.

The damping in pitch is found from the definition of k if M_w is known from wind-tunnel or static flight data,

while $M_{\dot{w}}$ results from the definition of b; thus,

$$M_q = \frac{k + UM_w}{Z_w} = \frac{k}{Z_w} - \left(\frac{C}{k_y^2}\right)\left(\frac{\partial C_m}{\partial C_L}\right)U$$

$$M_{\dot{w}} = -(b + Z_w + M_q)/U$$

where k_y = radius of gyration about y axis.

It will be noted that a value of M_w is required from other sources for the complete reduction of the oscillation data. Further developments may make even this unnecessary.

The results of an analysis (for 175 m.p.h. equivalent air speed and a center of gravity location of 26 per cent) are shown in Table II, where comparison is made with values estimated from theory and powered model tests. A reduction has also been made from data at 28 per cent and is included. An examination of this table shows that the test results are, in general, reasonable.

The significance of the above results lies in the demonstration of the feasibility of the new method in isolating aerodynamic derivatives, and other basic aerodynamic data. Some of these data are unobtainable from flight by other means, and the overall method and procedure for obtaining the derivatives is as simple as any that has yet been proposed. In fact; once b, k, etc., are known, the reduction to the aerodynamic parameters may be completed in less than an hour.

Obviously, much remains to be done before the method can be considered for routine engineering work. In the first place, a thorough examination of the reduction accuracies is being made to ensure that the present method is optimum. In the second place, it is proposed to reduce a large amount of flight-test data by the method, thus enabling checks and accumulation of experience.

The Step Function

In addition to the sinusoidal, there are many other forcing functions that may be of practical importance in stability and control work. Thus, for many years it has been customary to examine, for certain purposes, the result of abrupt manual control deflection. A further study of this technique, which will be referred to as the "step function," has been made and will now be discussed.

First, it should be emphasized that the sinusoidal forcing function has the great advantage of yielding a large volume of steady-state data, and it is likely that it will always excel other types of forcing functions in this respect. On the other hand, it has real disadvantages in the flight time required and the problems of data reduction— both of which may prove excessive for routine use. Also, there are many rapidly changing flight conditions (such as dive, or a maneuver at high acceleration) wherein time is not available for a steady-state examination, even if the human pilot would relinquish the flight controls completely and allow them to be operated automatically. (It is true, of course, that automatic control will be used more and more in these critical conditions of flight, but for the present they are approached with manually operated controls.) *The forcing function of short duration, with reduction of the necessary data from the transient response, therefore looms as an attractive possibility.*

A proposed technique will be discussed briefly. Thus, suppose that a pilot could, by pushing a button (located on the control stick), momentarily lock the cockpit control, while at the same time the control surface was displaced through a small angle (0.25° to 0.5°) relative to the cockpit control. Records of the airplane response in this brief period could be simultaneously obtained, and, if the information were sufficient and satisfactory, the imagined technique would be exceedingly useful in rapidly obtaining data in all sorts of maneuvering (and steady) flight conditions. Hydraulic and pneumatic mechanisms can easily be visualized which would make such a technique workable. In comparison to the sinusoidal method, it would be possible to simplify the instrumentation and eliminate the automatic pilot, which would be desirable for fighters and other "cramped quarters" airplanes. The overall time for the step application could be under the control of the pilot or adjusted to some predetermined amount (on the order of 0.5 to 0.8 sec.).

The establishment of the technique described above depends primarily upon its relationship to the mathematical step function, which assumes that the control is displaced instantaneously and that the corresponding applied aerodynamic moment acts in the same manner. By straightforward analysis the response to a unit elevator step $|\delta| 1(t)$, in terms of the dimensional b and k, may be expressed as

$$\frac{n(t)}{\delta} = \frac{UM_\delta Z_w}{k}\left[1 - e^{-(b/2)t}\sqrt{\frac{4k}{4k - b^2}}\sin\left(\frac{\sqrt{4k - b^2}}{2}t - \tan^{-1}\frac{\sqrt{4k - b^2}}{2}\right)\right]$$

$$\frac{q(t)}{\delta} = \frac{-Z_w M_\delta}{k}\left(1 - \sqrt{\frac{[b + (2k/Z_w)]^2}{4k - b^2} + 1}\left\{e^{-(b/2)t}\sin\left[\frac{\sqrt{4k - b^2}}{2}t + \tan^{-1}\frac{\sqrt{4k - b^2}}{b + (2k/Z_w)}\right]\right\}\right)$$

where, $b^2 < 4k$ (oscillatory)

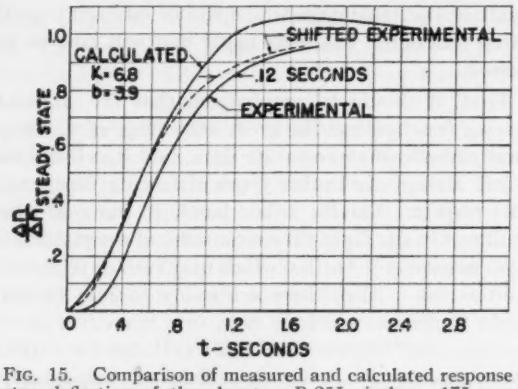

FIG. 15. Comparison of measured and calculated response to a step deflection of the elevator, B-25J airplane, 175 m.p.h. equivalent air speed, 10,000-ft. pressure altitude, center of gravity = 26 per cent M.A.C.

There was no assurance whatsoever that an equivalent of this mathematical step could be obtained in flight or even approached. Nevertheless, from a practical standpoint the possibility that an experimental step function could be used directly with the Duhamel integral to compute responses to arbitrary control motions (and that for certain cases good, if not better, agreement might be obtained than from a calculated step, based on experimental b and k) could not be overlooked. Obviously, a number of comparative checks would have to be made; thus:

(1) A comparison of a step function calculated from sinusoidal b and k data, with an experimental step.

(2) A calculation of b and k values directly from an experimental step, and a comparison with b and k from experimental oscillation data.

(3) A comparison with the results of an experimental arbitrary maneuver by (a) the use of b and k from sinusoidal tests and the Duhamel integral, and (b) the use of an experimental step and the Duhamel integral. If the experimental arbitrary maneuver results could be reproduced by either (a) or (b), then the validity of the particular technique has been established.

Experimental steps have been performed using both manual and automatic control. In most cases the elevator was snapped up about 1° to $1^1/_2$° from trim, as fast as possible manually, or at the maximum servo-output rate. In each case the control was displaced before any change in attitude of the airplane was recorded, although an extremely small momentary downward acceleration from δZ_δ was noted. From such tests as were made with other control surface displacements, it appears that for small angles the response is linear with displacement. From practical flight aspects the technique was entirely satisfactory.

A comparison as suggested under (1) is made in Fig. 15. The calculated curve was obtained by inserting measured b and k values in the step function response equation, given above. The experimental curve has been modified only by the application of electrical filter corrections, which in this case required a Fourier analysis. At the end of 0.8 sec., the airplane attitude had changed appreciably, and, since the speed had started to drop, the comparison is no longer entirely valid above this time. (With a "push down" it appears possible to reach an asymptotic value before appreciable speed change.) If M_δ is not exactly in phase with δ, as is assumed in the equations, and if, in addition, the b and k values from the sinusoidal reduction do not adequately take into account the effect of δZ_δ, a shift of the time scale of the measured curve may be required before a comparison is made. If the measured curve is arbitrarily shifted to the left by, in this case, approximately 0.1 sec., the agreement up to 0.8 sec. is then remarkably good, the general shapes comparing well and the slopes being identical in the range 0.4 to 0.8 sec.

A comparison of the time and effort required to obtain the two curves is interesting. The sinusoidal data for b and k took about 3 hours of flight, while securing the step data was a matter of less than 5 min. The time for record processing and data reduction is, of course, much greater for the sinusoidal case and is of the order of ten to one.

The step function results to date are inconclusive. Further tests are contemplated with a series of different step magnitudes (using improved filters) to establish empirically the initial shape of the curve. Meanwhile, methods of obtaining b and k from a step function response are being examined. The asymptotic value determines k if M_δ and Z_w are known; the extraction of b is more complicated. The practical aspects of the entire matter rest upon the arbitrary maneuver, comparison (3), which has recently been performed but as yet not analyzed.

In general, control characteristics (particularly hinge moment) are a function of the rate of application, and it is now well known that adverse effects may be obtained during rapid maneuvers. The automatic "step function equipment" enables a direct study of these effects by obtaining data at a number of predetermined control rates—i.e., this equipment may be used to apply the control at variable rates if desired.

Miscellaneous Developments

Speed Changes

Some further discussion of the constancy of forward speed during the oscillation tests is required. In Figs. 16a and 16b are shown measured longitudinal accelerations (properly reduced by removing the pitch angle component of gravity) and the indicated variations in speed from the air-speed indicator. (The latter are merely indicative of what the flight crew could observe directly.) Although the acceleration data are conclusive, extensive calculations have been made at both Massachusetts Institute of Technology[7] and Cornell to examine theoretically the probable speed variation over the frequency range. If values of b and k are computed from expressions that include all the U velocity terms

and are compared to the computed two-degree-of-freedom case, the following results are obtained (B-25J airplane, 175 m.p.h. equivalent air speed):

	b	k
Two degrees of freedom	3.78	4.284
Three degrees of freedom	3.77	4.252

Speed change effects may be taken as less than 2 per cent at frequencies as low as 0.5.

In connection with the effect of speed changes, etc., it should be noted that, for the cases where the experimental points fall off the circle diagram (at high or low frequencies), an empirical method has been developed for obtaining the effective b and k values, which are, of course, functions of frequency.

Hinge Moment Response

The hinge moment response may be expressed in relationships comparable to those for the control angle response, and effective damping and spring constants b' and k' may be found. A circle diagram of this reduction for a particular case is given in Fig. 12b, and the experimental variation of these parameters with center of gravity location is shown in Fig. 13b. It should be noted that the hinge moment data are obtained from strain gages at the control horns. Since servomotors in automatic pilots have limited torque outputs, the automatic pilot manufacturer is interested not only in the response of the airplane to control displacement but also to applied hinge moment; b' and k' fill this latter need. The automatic pilot may be thought of as a device that is neither "fixed" control nor "free" control, and the aerodynamic specifications for the automatic pilot must then include b, k, b', and k'.

From the aerodynamicist's standpoint, the hinge moment parameters offer additional relationships that can be used, as in the case of the derivative analysis, to isolate design factors of interest.

Special Sensing Equipment

In the course of the program a number of special sensing units have been developed or used. Among these are angle of attack, angle of yaw, indicated air speed, pressure altitude, and g sensing. The output from these sensing units is fed into the automatic pilot to maintain a given value of the variable sensed or to change (or rate of change) it at a predetermined rate. A combination of the g and speed sensing enables, in a single automatic maneuver, the procurement of all of the information desired for SF/g flying qualities tests, which has formerly been obtained (and only with considerable difficulty) in wind-up turns. A slow rate of change of control position device (for moving controls continuously and as slowly as desired) is of considerable use in obtaining static data from flight tests. Curves of static elevator angle, and stick force vs. speed, were obtained when this device was set to operate at the rate of 1° of elevator per 5 min. Further tests will indicate how fast a control may be moved before the assumption of static conditions is violated.

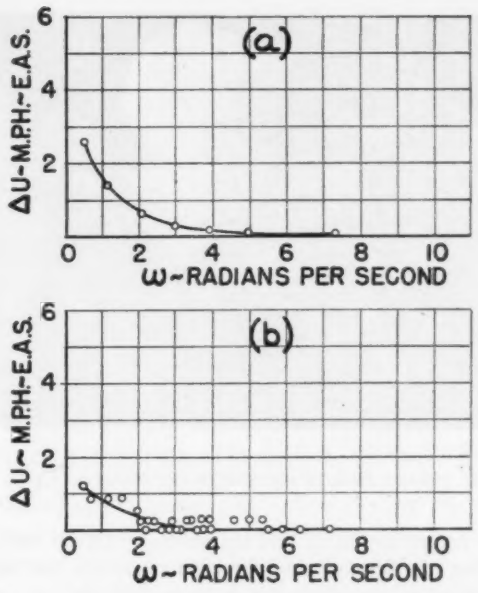

FIG. 16. Variation of speed during sinusoidal oscillations at several frequencies, B-25J airplane, 175 m.p.h. equivalent air speed, 10,000-ft. pressure altitude. (a) Calculated from longitudinal acceleration records measured in flight. (b) Observed from photo-observer records of air-speed indicator taken in flight.

Effect of Magnitude of Control Motion During Oscillation Tests

It was initially assumed in oscillation testing that the response was linear with control deflection, provided the deflections were small. Since it is desirable in some respects to use as large deflections as possible, experimental measurements have been made to ascertain the effect of changing this parameter. No effect on the response per unit of control deflection (up to 3.0° total deflection, the highest tried) was found.

Static Flight Checks

A number of static parameters (C_{L_α}, C_{m_α}, etc.) have been measured by well-known methods to check the information from oscillation tests. This type of checking has been a continuous part of the program and has been in part referred to in the derivative analysis.

Notes on the Instrumentation

Unlike many types of flight testing, a large part of the instrumentation for the present program has been especially developed. This instrumentation is largely electronic, and, while it is not the purpose of this report to go into detail, a brief discussion of some of the problems is given below.

Perhaps the major instrumentation problem to date has been that of filters. The necessity for adequate filters has been shown by a sample of unfiltered oscillo-

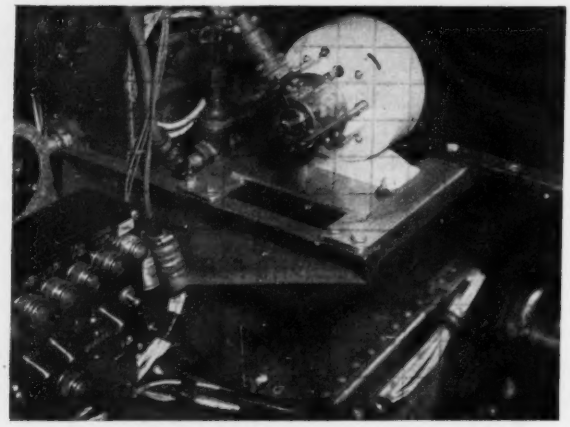

FIG. 17a. Slow rate of change of control position mechanism used with A-12 autopilot, B-25J airplane.

graph record in Fig. 2, and the necessity for an understanding or elimination of the filter responses has been discussed.

During the course of the tests, a phenomenon appeared in the records which has been referred to as the "beat." The oscillograph records showed, and it was visually confirmed in flight, that the airplane did not oscillate at regular amplitude but at even intervals made a cycle of higher amplitude. Although every effort was made to isolate the cause of this phenomenon, it suddenly disappeared and has never re-occurred. During its life, however, special instrumentation and modification of the oscillator and automatic pilot were required for its study, which posed original problems for the Instrument Section.

Fig. 17a shows the device developed for moving the elevator at extremely slow predetermined rates. This is an ingenious adaptation of an ordinary alarm clock. Fig. 17b illustrates the equipment for acceleration sensing and for applying step functions. The design

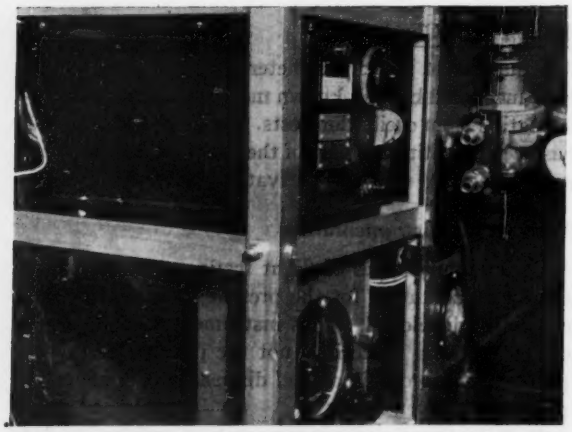

FIG. 17b. *Top:* normal acceleration sensing equipment. *Bottom:* step function equipment. Both used with A-12 autopilot, B-25J airplane.

requirements of the former were naturally stringent to avoid any possibility of exceeding allowable acceleration limits and, hence, have structural failure of the airplane. The device incorporates a unique g limiting device that can be present to a safe allowable value. With appropriate servos, the step function equipment can be used to obtain step functions of any magnitude. In operation, the control button is pushed when the necessary flight conditions have been established. The control snaps over and remains deflected until the button is released.

Instrumentation development, calibration, and maintenance for a research of this type must be looked upon as a major part of the operation and the program.

Weight, Center of Gravity, and Inertia Control

An important practical consideration in this type of flight testing is that of the control of weight and its distribution. Early in the program, water transfer provisions were made so that the c.g. could be shifted in flight. A tail water tank (400-lb. capacity) allows for a shift of 5 per cent M.A.C. at normal gross weight. The water in this tank can be controlled to ± 10 lbs., and the c.g. may be placed in flight to a precision of 0.2 per cent M.A.C. The weight at a particular time may be ascertained to about 1 per cent.

Basic moment of inertia data as supplied by the manufacturer had been used with appropriate corrections for configuration changes. Calculations revealed that large moment of inertia changes (on the order of 30 per cent for the pitch axis) were encountered on shifting water ballast for c.g. control. Under these circumstances, in consideration of the dependence of the response measurements on moment of inertia, a study was made of methods of experimentally determining the moments of inertia. *It was concluded that the method of swinging, which has occasionally been used, suffers inaccuracies unless care is taken to use the optimum pendulum length for each case.* The practical problems connected with locating a supporting structure of the correct height and swinging an airplane of the size of the B-25 led to the development of a ground oscillation technique. The airplane was pivoted at the wing-jack points, while supported at the nose gear location on a set of coiled springs of known characteristics. Records were taken on the oscillograph of the time history of the pitch angle during a free oscillation. For practical purposes it was found possible to neglect the damping in the system, since several cycles of oscillation would be obtained with little change in amplitude. This facilitated the determination of the longitudinal moment of inertia from the data. The complete results of these tests were not available in time for inclusion in the present report.

Consideration is also being given to the development of techniques for the experimental determination of the moments of inertia about the X and Z axes of the airplane for the lateral program.

Part IV—Future Developments

Dynamic Lateral Stability and Control

A program of dynamic lateral exploratory tests has been completed on the B-25 airplane. These tests, which utilized automatic control, involved about 20 hours of flying. In addition, extensive plans, theoretical developments, and the preparation of special instrumentation have been under way for the complete lateral program.

In general, the philosophy of the lateral program has been that, once the longitudinal motion was thoroughly mastered, the technique and experience could be transferred to the lateral case. In this case, involving as it does the unsymmetrical motion of the airplane, automatic control offers possibilities in constraining a particular velocity component, resulting in an effectively simpler motion for study and analysis. Thus it is apparent that the sideslip velocity might be maintained at zero through the use of a sensing device (controlling the rudder through the automatic control) based on yaw angle and/or lateral acceleration and, if necessary, their derivatives. Such constraint would reduce the motion to that of a "two-controlled" type (such as would be experienced with an airplane of high static directional stability and damping), since the sideslip velocity would be maintained approximately zero at all times. Many possible types of constraint, of which that described is but one, may be visualized for the lateral motion by the appropriate adaptation of automatic control.

The exploratory tests have also included examination of a wide variety of suggested oscillation techniques, with, in each case, the primary objective of establishing their feasibility from the practical flight operating standpoint. While space does not permit a complete review of these many tests, it can be mentioned that provision was made for oscillating the rudder and aileron separately or in combination. These control oscillations could be performed with various functions of the automatic pilot in effect. Thus, the control oscillations could be superimposed upon the normal automatic pilot function (which is that of maintaining an overall trim attitude), as in the longitudinal case, or, at the other extreme, the servomotors could be made to oscillate the controls with no normal automatic pilot function present. Further provisions were made for utilizing various amounts of gyro signal (adjustable in flight) in the rudder and/or aileron circuits. The recording instrumentation was maintained at a minimum during the exploratory tests, although a small photo-observer was utilized for the specific purpose of establishing that the aileron and rudder motions were satisfactorily sinusoidal during the oscillations. The various test conditions were set up largely through visual instrumentation of sideslip angle, side acceleration, etc.

The theoretical work performed simultaneously with the exploratory flights was directed at an examination of the various techniques under test, with an eye to establishing the lateral information that each would yield, the specific recordings that would be required, and the methods of data reduction and analysis.

As a result of all of this exploratory work, the scheme that offers the most promise, both from a flight and analysis standpoint, may be described as follows:

Provision is made to oscillate both the ailerons and the rudder simultaneously and at identical frequencies by means of the automatic controlling system. In the case of either the rudder or aileron control, the normal automatic pilot function (gyro signal) is introduced in an amount necessary to maintain average trim attitude during the oscillation. The control motions are limited to small amplitudes, which allow the ensuing lateral motions to be considered as independent of the longitudinal motions. (The elevator remains on normal automatic control throughout.) Provision is made for the manual adjustment of the phase between the two control motions and their maximum amplitude ratio. By an appropriate adjustment of the phase and amplitude ratio control, it is possible to show theoretically, and has also been proved experimentally, that three special lateral maneuvers may be established—each maneuver being defined by the elimination of one of the lateral velocity variables, that is, sideslip velocity, rolling velocity, or yawing velocity. Thus, the first maneuver, in which the sideslip velocity $v = 0$, consists of a series of perfectly coordinated turns; the second maneuver, in which the rolling velocity $p = 0$, consists of a yawing oscillation in which the wings remain level; while the third maneuver, in which the yawing velocity $r = 0$, consists of a rolling oscillation at constant heading. Each of the maneuvers is repeated, as in the longitudinal case, for the entire usable range of frequencies (0 to 8 rad. per sec.), and an oscillographic recording is made of all of the pertinent variables of the motion, such as the angular displacements, the angular velocities, linear and angular accelerations, etc., as well as the phase angles of the "forcing" control motions and the amplitude ratios.

The theoretical analysis indicates that, from the data obtained through tests of the above type, it should be possible to reduce, in pairs, all of the lateral stability and control derivatives—both static and dynamic. It may even be possible to obtain all of this information from two of the maneuvers, or it may be possible to obtain duplicate information on various derivatives if all of the special maneuvers are used, thus enabling an extremely valuable experimental check. The reduction of the data can proceed upon semigraphical lines, after the manner of the longitudinal analysis and, hence, enable the determination of effective spring and damping constants of those modes of the lateral motion which are of practical import.

This "double oscillation" lateral forcing function technique, which represents such a powerful attack on the lateral case, was first suggested by J. B. Rea while fulfilling the requirements for his Doctorate at the Instrumentation Laboratory, M.I.T. The development of the flight technique is the work of the Flight Research Department at Cornell Aeronautical Laboratory. Advances in the method for extracting the individual lateral derivatives from the test data are the result of the joint efforts of I. G. Ross, W. O. Breuhaus, and D. W. Whitcomb, of the Flight Research staff.

To complete the lateral picture, it is likely that the step function technique will be utilized, and, in addition, static flight checks will be performed by standard "applied moment" methods.

In judging the significance of a flight method for the extraction of all of the lateral derivatives, attention should be directed to the fact that at the present time there is no commonly used method, either wind tunnel or flight, for obtaining the damping and coupling terms. Furthermore, there is no flight method available for obtaining directly the magnitudes of the control effectivenesses and the important static lateral derivatives, such as the effective dihedral and the directional stability.

Combined Longitudinal and Lateral Motion

Flight measurements of the combined dynamic longitudinal and lateral motion are almost nonexistent, although a considerable fund of static information on such items as the pitching moment due to sideslip is available. When automatic control is used for rapid maneuvering, such as tracking, considerable interest is attached to the entry and recovery from turns and the characteristics in steady turns with and without sideslip. The longitudinal characteristics, thought of alone, may be applicably modified by the steady increment of normal acceleration, which maintains in steep turns or rapid pull-ups. Furthermore, the motions resulting from rough air are a combination of the longitudinal and lateral modes, even though the airplane is attempting to proceed along a steady rectilinear flight path.

Exploratory tests have been completed which demonstrate the practical feasibility of performing elevator oscillation tests under accelerated conditions in turns of various degrees of bank. Other tests have been run which show that similar tests may be made with various degrees of initial yaw. Finally, by a combination of the oscillator control and automatic pilot turn control, oscillation data are available during turn entry. To survey and isolate the critical conditions for combined longitudinal and lateral motion will be a major research effort in itself.

Long-Range Dynamic Research Program

A program for the continuance of the research in basic dynamic stability and control (which is in accordance with the initial objectives of Part I) is mapped out in preliminary form below. Such a program would represent in the neighborhood of 5 years of research and in the author's opinion is a logical one for furthering the cause of dynamic analysis so that its import will be effectively felt in several allied fields of design. The work outlined in Phase I is now largely planned or in work under the sponsorship of the Air Materiel Command. In Phase II are collected those items that should follow immediately upon the completion of Phase I. This phase is largely concerned with the application and "pilot" trial of methods developed in Phase I and is a necessary and important phase in any broad research. Phase III is concerned with specific and currently foreseeable dynamic problems of sufficient complexity and importance to warrant particular research. It can be anticipated that in this phase all the tools of previous research would be effectively utilized.

Phase I

(1) Perfection of longitudinal methods with particular attention to step function.

(2) Adaptation of methods to lateral and combined longitudinal and lateral motion.

(3) Development of sideslip, roll, etc., sensing equipment as required by (2) above.

(4) Study of application of the longitudinal oscillation technique to high-speed (subsonic) airplanes and to sub-, trans-, and supersonic missiles.

(5) Study of application of oscillation data to specific problems in automatic control and simulator design.

(6) Study of application of advanced electrical engineering methods to aerodynamic stability and control problems.

Phase II

(1) Longitudinal oscillation tests on a series of widely different types of airplanes for the purpose of amassing experimental information over a wide range of design variables.

(2) Simplification of longitudinal oscillation technique for routine use and for (1) above.

(3) Repeat (1), using best lateral technique.

(4) Study of information from (1) and (3) above in reference to (a) the specification of dynamic flying qualities characteristics and (b) the specification of commercial and military airplane behavior on automatic control.

(5) Application of the longitudinal oscillation method to a high subsonic speed airplane (P-80 class) on the basis of Phase I (4).

(6) Longitudinal oscillation tests on supersonic missiles with telemetered information.

(7) Extension of longitudinal oscillation technique into the unstable range (on a B-25J class airplane).

(8) Development of electronic simulator for utilizing b and k for routine use in the research.

(9) Development of dynamic design charts for the new parameters, k and b, and in nondimensional form.

(10) Study of rough-air behavior of airplanes on automatic control.

Phase III

(1) Research on practical rough-air control equipment.

(2) Stall research using automatic control and the development of stall control.

(3) Special missile applications.

(4) Research in the measurement of the dynamic characteristics of helicopters.

General Conclusions

Substantial progress has been made in the attainment of the objectives laid down for this research:

(1) New methods and techniques have been developed which have wide application in the measurement of dynamic stability and control in flight.

(2) By the use of these methods, a large amount of full-scale dynamic information has been obtained on service-type aircraft for the longitudinal motion.

(3) Substantial progress has been made in simplifying the analytical approach, particularly in reference to aerodynamic data for automatic control design and in enabling rational reduction of the flight information.

(4) Some insight has been obtained into the range of validity of the classical assumptions, and an approximate method for determining the aerodynamic derivatives from flight has been demonstrated.

(5) The use of automatic control for this type of work has been thoroughly established, as well as its attributes in other types of flight testing.

(6) A tremendous amount of experience in all phases of this type of research is now available to be directed at the rapid attainment of the remaining objectives.

NOTE: *More detailed discussion of both the theoretical and experimental aspects of this research has recently been published in a series of Cornell Aeronautical Laboratory technical reports.*[88–91] *For a résumé of lateral response theory and advances in the longitudinal treatment, reference is made to these reports.*

Acknowledgment

Appreciation is extended to those many individuals in the sponsoring and other agencies who have, by their enthusiastic support, made this research possible.

The author wishes particularly to acknowledge the assistance of W. O. Breuhaus, and G. F. Campbell, J. P. Chawla, R. C. Kidder, R. L. McManus, and R. C. Sager of the Flight Research staff, in the preparation of this report.

These acknowledgments would not be complete without a general expression of appreciation to Otto C. Koppen, who has inspired so much research in dynamic stability and control.

Appendix—Synopsis of Previous Dynamic Research

Up to about 3 years ago, at the time of the initiation of the present work, progress in dynamic stability and control in the English-speaking countries was largely associated with conventional airplane development. Although in Germany and, more recently, in other countries emphasis has been placed on dynamics in missile and pilotless aircraft application, it is currently difficult to evaluate these efforts from the standpoint of fundamental progress. A completely digested picture of the German work is not yet available, and a general review is still hampered by existing classification. A large number of references to basic German dynamic investigations, none of them translated and many not readily available, have come to the author's attention. A large portion of these relate, interestingly enough, to combined aerodynamic and automatic control problems. The hopelessness of evaluating German progress in this field at the present time is apparent. Some notes on interrogations arranged through Foreign Information, T-2, Air Matériel Command, are, however, included at the end of this Appendix.

A sufficient number of key or typical references are given below to enable the reader to pursue a particular phase of the literature. No attempt is made at a complete bibliography.

Classical Theory and Early Model Experiment

Bryan's pioneering work in aircraft dynamics, based on the rigid dynamics of Routh, was initiated in the year of the airplane's birth (1903), and its complete presentation in *Stability in Aviation*[8] in 1911 marks the beginning of the so-called classical stability approach. The complete mathematical statement of the fixed control motion of an aircraft was presented in six simultaneous linear differential equations with constant coefficients, predicated on the assumption of small disturbances from steady symmetrical flight and with component air forces as linear functions of the velocities (i.e., the air force derivatives). These linearized equations are written for axes that are fixed in the airplane. For the reason of airplane symmetry and small disturbances, the equations normally separate into two groups—the longitudinal and the lateral. It is these equations, without fundamental change, that have for the last 35 years represented the most common approach to the study of airplane stability. Almost the sum total of the mathematical and experimental work in aircraft dynamics has therefore evolved about equations whose rigorous application does not extend beyond small departures from steady trimmed symmetrical flight and which are incapable of handling problems of finite maneuvering in curved flight—a limitation, however, nowhere near so drastic as it might first appear. The earliest, and one of the few, exceptions to Bryan's approach was that of Lanchester,[9] which depends on great simplifying assumptions, largely prohibiting its use in research or design.

In the years following Bryan's original presentation, to about 1920, much effort was expended in further mathematical refinements and generalization of this theory. In Great Britain, contributions were made by Bairstow, Jones, Glauert, and Brodetsky,[10–12] while in this country Wilson,[14] in some of the earliest N.A.C.A. work, studied the motion of an aircraft encountering gusts, pointing out among other things the significance of the short period to acceleration response. By 1919 Glauert had published a complete theoretical treatment of the free-control longitudinal stability[15] and the stability derivatives of the propeller,[16] while Bateman had introduced a form of the stability diagram.

In parallel with the theoretical was the comprehensive development of wind-tunnel methods and techniques for the measure-

ment of the individual stability derivatives. While this work originated in Great Britain, it was introduced into this country in about 1915 by Hunsaker, who, with others, ran complete dynamic tests on two models in the old 4-ft. rectangular tunnel at the Massachusetts Institute of Technology. These results are presented in a Smithsonian Report, *Dynamic Stability of Aeroplanes*,[17] perhaps the best known early presentation of the subject in this country and one that can hardly be surpassed as a basic reference today. The work was continued by Warner, Klemin, and others with increased emphasis on the effect of design changes. In Great Britain, the refinement of model methods for the determination of the individual derivatives continued, though at a decreasing pace, throughout the 1920's. Glauert, Bryant, Irving, Relf, Bradfield, Halliday, and others[18–21] accomplished an amazing amount of research with continuous rotation balances and simple (one degree of freedom) oscillation techniques in the wind tunnel and with the whirling arm. Only a small typical part of this work is cited in the references.

The introduction of a nondimensional form of the stability equations by Glauert[22] in 1927 was a first step toward their engineering use, and it facilitated the examination of the effect of primary variables upon the motion. Based on these equations, Gates[23] surveyed the unstalled longitudinal motion, presenting the results graphically in stability diagrams. It is doubtful, however, that even this simplified presentation was greatly used in design.

Complete texts of the progress of dynamic stability and control were available for the student in 1920 in a paper by Glauert[24] and in Bairstow,[25*] *Applied Aerodynamics*, and Wilson,[26] *Aeronautics*, which were followed by Warner,[27] *Airplane Design*, in 1927. A review of progress at this stage indicates that, while the Bryan classical approach had become well established in research and model methods were available for experimental determination of the derivatives, its use in practical design was negligible. In fact, the static approach, utilizing only the static derivatives, had become the recognized design tool. Some of the shortcomings of the dynamic approach, such as its almost complete confinement to the determination of the period and damping of the phugoid and the prohibitive time for a complete wind-tunnel test of a particular model, were well recognized. In later years, even in research, the specialized balance and whirling arm equipment became less and less used, for it was not adaptable to the determination of information at higher Reynolds Numbers. Other seemingly more profitable avenues of research led to a change in emphasis as the picture developed.

Initial Flight Measurements and Stall Studies

Almost as soon as wind-tunnel methods were proposed for the determination of the derivatives, attention was directed to their measurement in flight. Preliminary dynamic flight tests were performed by the British prior to 1920,[28] but the outstanding work of the early period is that of Norton, Warner, and Allen,[29–33] at the N.A.C.A., which took place between 1919 and 1923. It is interesting to note that this work marks the beginning of N.A.C.A. flight-test and instrumentation development and E. T. Allen's debut as an engineering test pilot. In the course of this program, a number of the derivatives were determined (l_p, l_v, n_r, y_v, m_w, etc.), and use was made of the "step function" (sand dropped from wing tip) and the "known applied moment" (parachute at wing tip) techniques. Consideration was given to desirable flying qualities in relationship to basic stability and control, and comprehensive measurements were made of accelerated flight.

By 1925 the first of the several reports on the full-scale determination of the derivatives for a Bristol Fighter, BF2b,[34] was published by the British. These tests required several years to complete and are unique in that at no time since have such complete measurements of the lateral derivatives been accomplished full scale. With the exception of the determination of the longitudinal derivatives for a light airplane at the N.A.C.A.[35] and isolated tests in Germany (Dornier), and elsewhere, no further flight tests for the complete dynamic derivatives were made. Although the agreement in unstalled flight between theory, wind tunnel, and full scale was at that time (14 years ago) reasonable—particularly for power-off conditions—the extended use of classical dynamics in research and design has been based on little more full-scale flight proof than that noted.

A dynamic problem of great complexity is that of the stall. In 1920, coincidental with research in the unstalled régime, the British launched a determined attack on this problem, making use of all wind-tunnel, theoretical, and flight resources (with results that were even less known and used by the practicing engineer than those from the unstalled investigations). In the course of this work, extensive calculations were performed in the step-by-step integration[36] of the equations, thus enabling the motion to be followed for arbitrary control displacements.

Tests[37–39] were performed in stalled glides at angles of attack of up to 50° which indicated the superiority of the large vertical tail at the stall, the effectiveness of slots and interceptors as control, etc. While this overall program[40] gave a basic understanding of stall dynamics, much remains to be learned. Although the British assumed the existence of the conventional velocity derivatives, it now seems apparent that the classical approach is inadequate for completely explaining even the initial divergence. In this country, with the exception of Koppen's investigations, the problem of control at the stall has largely been thought of in terms of basic wing flow breakdown. The problem is a dynamic one, however, involving the aircraft as a whole and is becoming more important with extreme plan form and airfoil shapes, which for performance reasons can be subject to but limited modification.

Research in the Decade Prior to the War

Since about 1930 the course of airplane dynamics in this country has been widely and continuously influenced by the research and teachings of Otto C. Koppen.[41] His theoretical investigations, wind-tunnel work (oscillators), and achievements in airplane design are now well known. For years his course in stability and control at M.I.T. was unique in its treatment of the complete dynamics, and many current trends in design and research may be directly traced to his work as an educator. The elevator forcing function technique, with which the bulk of the work of the present paper is concerned, had earlier been suggested by Koppen, and it was through his influence that the investigations were initially directed at the longitudinal motion.

In 1935, Mueller[42] devised one of the first simulators, an electrical device for solving the classical longitudinal stability equations. Unfortunately, a similar device for the lateral motion was never constructed, nor was the effect of controls introduced into the longitudinal machine. Probably the first application of the electrical engineer's rotating vector diagram to the solution of airplane stability problems was presented by Mueller[43] in 1937. This approach deserves great attention, allowing, as it does, a strobotactic view of the elements of a vibration problem and their phase relationships. In a section of the present report, a similar "electrical" method has been applied to the determination of effective damping and spring constants from experimental flight data, an application suggested by I. G. Ross. In a later publication, Mueller summarized the determination of dynamic data from scale-model airplanes.

An extended series of wind-tunnel tests was initiated by the N.A.C.A. in 1932 for the evaluation of lateral control devices.[44] Subsequent research indicated that the quasi-static rolling criteria used were inadequate, and a method based upon a step-by-step

* The second edition, 1939, of this text is widely used as a basic reference. The dimensional notation is used. Bairstow's original work in stability was so prominent that the basic approach is often referred to as that of "Bryan and Bairstow."

solution of the classical equations was devised by Jones and Weick,[45] 1936, and applied in a final evaluation[46] in 1937. Paralleling the wind-tunnel research, an intensive flight program, utilizing various devices and aimed principally at control effectiveness, was initiated on an F-22 airplane. These remarkable flight tests under Weick, Gough, Soule, McAvoy,[47] and others, led to first-hand knowledge of spoiler lag, two-control operation, effect of excessive dihedral, etc. The experimental methods themselves were widely copied by the industry. Extensive tests of the F-22 in the full-scale tunnel were made in conjunction with the flight program. The backlog of experience from these tests appears to have been influential in Gilruth's later flying qualities developments.

Zimmerman,[48] in two widely known N.A.C.A. reports, presented a complete engineering analysis of the power-off longitudinal and lateral stability, with charts for design use. The approach was entirely classical and followed in part the earlier work of Gates and Koppen but with an original and comprehensive examination of the phugoid. They brought to the attention of the American engineer the great advantage of the nondimensional notation. In addition to Zimmerman's work, the N.A.C.A. continued its theoretical and wind-tunnel researches into the evaluation of the individual derivatives, and by 1942 a large mass of information was available to the designer for the estimation of these constants under normal flight conditions. Reference[49, 50] is here made to but a typical part of this expanding literature. Many specific applications, such as to two-control operation, were also considered.

As early as 1930 Bryant and Williams[51] had suggested the use of the method of operators for calculations of the disturbed motion of an airplane. A simplified, and now widely used, application of the method was given by Jones[52] in 1936, who later followed with publications on allied research. The method has since seen considerable use in the design of unconventional aircraft (such as flying wings), in the computation of maneuver loads on tail surfaces by Perkins[53] and others, and in automatic control applications. An improvement in the graphical solution of the Duhamel integral has recently been suggested by Laitone[54] and is specifically referred to in the body of this report. The introduction of the method of operators to dynamic stability and control calculations represents one of the few advances made in the mathematical treatment of the subject since the time of Bryan.

With the exception of work referred to elsewhere, European progress in dynamic stability and control in the 10 years preceding the war was not great. In 1930, as the result of directional hunting on Schneider Cup racers, the British investigated[55] the directional stability of high-speed aircraft from a purely classical approach, a work significant in the fact that similar difficulties have since been encountered on dive-bombing and fighter aircraft and have since been more thoroughly investigated. The effect on stability of high-wing loadings had been estimated through the medium of the nondimensional equations, and attention was being directed to the behavior of unconventional types (tailless, autogyro, etc.). In 1934, Jones's[2] review of progress was presented in Durand's *Aerodynamic Theory*, which is still the most widely used reference text. In the late thirties, the trend of British research, subjected to American influence, swung far toward application, with emphasis on statics. German research had, of course, accelerated at an unprecedented pace.

A powerful experimental method of studying dynamic behavior came into common use in Great Britain in the early thirties. This was the testing of dynamically similar models, which is basically older than the airplane itself (Langley's $^1/_4$-scale aerodrome). Direct observation of these models gave information that could, through experience, be correlated with full-scale behavior. In recent years the method has been applied to the hydrodynamic characteristics of seaplanes by Coombes, Perring, and Johnston[56] and introduced into this country by Stout, Shoemaker, Truscott, and others. The preliminary tests in the N.A.C.A.'s first free-spinning tunnel were presented by Zimmerman[57] (1936), who later originated the free-flight tunnel.[58] At about this same time the N.A.C.A. gust tunnel for studying dynamic loads was placed in operation. While the hydrodynamic porpoising and the free-spinning model tests have substantially replaced, in their respective fields, other methods of design, it is unlikely that the free-flight and gust tunnels will ever reach such a degree of sufficiency, dealing as they do with problems throughout the flight range. In its present form the free-flight tunnel has demonstrated its remarkable capabilities in coping with a wide range of dynamic problems[59] (on conventional, unconventional, powered, stable, unstable, and towed models), the reference cited being but typical. Though checks have been made with the full scale, its limitations remain as those of scale, small deviations from straight flight and the physical size of the model which pose serious model design problems. In addition to the free-flight model, many proposals have been put forth for semiconstrained models, such as a model attached only to a drag wire. No outstanding results with this type of constraint are known.

One step removed from the free-flight tunnel is the free-flying radio, or otherwise remotely controlled, model exploited by Zimmerman and, more recently, exhaustively developed by Stout,[60] which seems eminently suited to three-dimensional maneuvering problems. A further development of the flying model is the use of small man-carrying aircraft to simulate larger, heavier, or higher performance machines, a scheme that has been used with considerable success by Short Brothers (Great Britain), Martin, Boeing, Chance Vought, and others, with wide variations in the degree of similitude. This technique of flying man-carrying models has come under discussion relative to missile dynamics.

In the early thirties, attention was directed to the problem of the desirable flight characteristics of the manually controlled airplane, a difficult phase of the picture since it involved a consolidation of practical pilot experience with engineering fact. This had never previously received adequate attention. In about 1935, E. T. Allen observed that large German aircraft possessed superior handling qualities, a trend that seemed hardly fortuitous. Soule[61] (1936) compared the longitudinal dynamic stability of several airplanes with pilot observations, which finally eliminated the phugoid from consideration as an important flying quality, and about the same time Warner set down the handling specification for the original DC-4. By 1940, Soule had reported on his preliminary flying qualities investigation (an outgrowth of the DC-4 specification), and the work was well under way with Gilruth's and Turner's work of the same year. Gilruth's[62] *Requirements for Satisfactory Flying Qualities of Airplanes* (1941) became a basic reference on the subject. Following this work, flying qualities requirements have been promulgated for military airplanes through the efforts of Perkins,[63, 64] Abzug,[65] and others, and complete tests have been performed on over 30 American and foreign military aircraft of all types. This impressive array of full-scale information has gone far in orienting effort in the stability and control field. Further ramifications have been studied by Phillips,[66] Kaytan, etc., and the entire subject (with test methods) was reviewed with the industry and services in a series of conferences in October, 1944, at N.A.C.A. Similar specifications are now available from British and German experience. In all of these specifications, a number of dynamic and quasi-static requirements are included which may be modified or appended as new experience and methods become available. The serious development of specifications like these for civil aircraft use may result in stringent requirements relative to rough-air behavior and performance on automatic pilot. The entire subject of desirable flying characteristics is, of course, directly related to research in dynamics whose function is to further the attainment of information that will enable the design of aircraft for predetermined characteristics.

A flight method for the numerical determination of the slope of the static pitching moment, though not required by the flying

qualities specifications, has been developed to a point of practical utility for comparing wind tunnel with full scale, and in specifying safe operating c.g. limits. This method, most widely used by Boeing,[67] yields total moment change with speed, hence having dynamic implications. Known as the "effective weight moment method," it is not new in principle.

A new tool for the measurement of the derivatives, and the study of dynamic phenomena in general, is the N.A.C.A.'s stability and control tunnel. By placing the model in a curved or rotating air stream, many of the disadvantages of previous dynamic balances are avoided, and it should be possible to conduct tests at higher Reynolds Numbers. It is understood that no dynamic data are yet available from this tunnel.

By 1940 a modern and comprehensive analysis of free-control stability was completed by Jones and Cohen.[68]

Recent American and British Trends

During the war and up to the present time, the major effort in dynamic research has been in the direction of application. This effort, largely based on established methods, has served to reduce partially the reluctance of designers to make use of the fund of dynamic information already available. (The experience of A. C. Reed and the author, in connection with the design of a prone position flying wing fighter, is illustrative of the necessity for resorting to fundamental calculations of the motion, since the magnitudes of the static stabilities were not only different from conventional designs but could not be judged in terms of conventional design experience. Others engaged in the design of unconventional aircraft have faced like problems and have made computations of the motion resulting from control displacement by step-by-step integrations, Duhamel's integral, or straight solution.) Among the many practical problems to which research has been directed during the war years are the damping of the lateral oscillation, aileron roll and stick force characteristics, the effect of friction in controls, wing-fuselage interference effect on the derivatives, and power effects, to cite but a few. The large amount of applied dynamic research still in progress in the large research laboratories defies ready classification.

In addition to applied research, some attention has been directed to more fundamental dynamic considerations. A portion of this work relates to the determination of the derivatives (for small disturbances) for various types, and degrees of rapidity, of motion. It will be appreciated that for extremely slow motion the simple Bryan derivatives depending only upon the present value of the velocities are probably satisfactory; on the other hand, for a motion of the rapidity of flutter, the aerodynamic forces do not follow such simple laws, and resort must be had to theoretically calculated flutter derivatives. It is now believed that even for motion in the vicinity of the longitudinal short period, the simpler form of derivative may be inadequate. This entire subject, linked as it is with flutter research, represents a broad field for investigation for many years and is mentioned in connection with the investigations of the present report. In relation to the domain of dynamic stability and control, reference is made to the work of Temple, Jones, Duncan, Cicala, Reid, and Vincenti.[69–73] Furthermore, it is becoming clear that for certain wing configurations, such as high sweep, it is no longer possible to estimate even the simplest type of derivative from present aerodynamic theory. Effort in both Great Britain[74] and this country is being directed toward further span load analysis.

Since about 1940, Mitchell, Gandy, Bryant,[75–77] and others in Great Britain have labored upon the calculation of airplane response characteristics for wide ranges of design parameters. The case for these extensive calculations has been well stated by Bryant,[74] Gates,[78] and Mitchell and Thorpe.[79] The response calculations have been made for various types of applied disturbances, usually the simple step. The calculations were performed on the differential analyzer, which represents the first time this tool has been put to such extensive use in the interests of airplane dynamic stability and control.

Application of the differential analyzer is representative of a general trend toward calculators and simulators of all sorts. The situation regarding these various devices in stability and control research has recently been summarized by Blaschke.[80] Since in all cases the motion is expressible in some type of differential equation, the advantage attendant to the use of such devices in stability and control is obvious. The reference alluded to[80] not only examines the principles of available machines but contains a complete bibliography of the literature. The use of such equipment should go far in removing the stigma that has so often dogged stability calculations in the past; however, the main problem remains that of providing the basic aerodynamic data. The devices now under examination vary from simple computers to elaborate "Link Trainer" types into which arbitrary control motions may be continually fed. In addition to machines, research has been directed to the solution of the differential equations by tabular, matrix, and graphical means.

An electrical balance has been proposed by Bratt and Wright,[81] the measuring elements being of the Eastman type (University of Washington wind tunnel) in which the support reactions would be measured during forced oscillation of the model. The balance should have utility in both stability and flutter work. (The author, in 1942, proposed a strain-gage type of dynamic balance for the high-speed wind-tunnel measurement of derivatives on small models. The realistic design of dynamic balances for high-speed work should, it is now believed, follow further dynamic flight tests.) The amount of dynamic research performed in high-speed tunnels appears to be extremely small.

In the more generalized motion of the airplane in which the longitudinal and/or lateral motions are of finite magnitude, some progress has been made. This has been summarized by Brodetsky[82] in a contribution extending the earlier work of Lanchester and Hopf. Of course, the finite motions of airplanes have been considered in specific problems (landing flare, dive pullout, stall, etc.) for many years, but in these cases resort was invariably made to step-by-step integrations or drastic simplifying assumptions.

Some results have been obtained in both England and this country in the measurement of atmospheric turbulence, which is a necessary preliminary to the dynamic design of airplanes with improved rough-air flying characteristics. Progress has been made in the use of airplanes as gust meters and will be enhanced by better methods of specifying the basic smooth-air characteristics. In the author's opinion, the need for rough-air response measurements and atmospheric turbulence studies is well summarized by Gates and Perring.[83] This is a large and important field for research.

Automatic Control Research

It is not within the province of this report to examine the background of design and analysis used in the automatic control field other than to emphasize again that a lack of engineering data on airplane dynamics has been a limiting factor in rational design. In general, it appears that the many automatic control studies that rest on the classical airplane stability equations have been of little use in design; conversely, most proposed simple analyses have failed to approximate the aerodynamic facts. In this dilemma the automatic control designer has proceeded largely on an empirical course with a minimum of professional aerodynamic assistance. With his electrical background, he has frequently had, however, a better understanding of the dynamic characteristics of a particular airplane than its own design staff.

Résumé of German Advances from Discussion with German Scientists

From discussion with German scientists formally engaged in stability and control research for airplanes and missiles, it appears

that important advances have been made in the general handling of aircraft dynamics, particularly in connection with automatic control where refined methods were in general use. It was the practice in the larger aircraft firms to perform complete dynamic analyses for each new aircraft.

The basic approach to dynamic problems was essentially classical. As a result of a comparison of the different reference axes by Rautenberg, it was concluded that for many purposes axes fixed in the flight path were most desirable. The limitations of the derivatives at higher frequencies were recognized. The restrictions in wind-tunnel measurements of the derivatives were well recognized and other methods of research were sought, one of these being a whirling arm submerged in water. With this device, sizable "aerodynamic forces" could be obtained in comparison to the centrifugal forces involved which should improve the measurement accuracy over the conventional (air) whirling arm. A complete dynamic strain gage balance was designed (Thiel), but it was not built when calculations of its basic accuracy led to the belief that the tares would be excessive. Sufficient wind-tunnel measurements of N_r with running propeller were made to ensure the accurate calculation of this derivative. Research directed toward the experimental measurement of N_p on the rolling balance has been reported by Thiel.

As in other countries, only a small amount of full-scale airplane research was performed (other than in connection with automatic control design). Some of this work has been reported by Schmidt and Stiess, was done on a Dornier DO-117, and was largely confined to the static lateral derivatives Y_v, N_v, L_v. Other flight measurements have been reported by Wenk and Doetsch of D.V.L. Insofar as the author could determine, no sinusoidal control forcing function tests were made in flight, the closest approach to this technique being that used by von Schlippe (Junkers, Dessau) in flight flutter tests.

In the field of automatic control, many unique possibilities were investigated. Among these were the possibilities of automatic control in counteracting the instabilities found at high Mach Numbers, the improvement of rough-air flying characteristics, and the increase in damping about a particular axis of an airplane otherwise flown manually. Some typical references to the automatic control dynamic work are given,[84-87] it being appreciated that these are chosen almost at random from several hundred allied reports.

Sufficient experimental work was done with the stall to indicate its primary dependence on angle of yaw and rolling velocity and to determine that these would be controlling variables for automatic control of the stall.

Control step functions, or "personal gusts," were used in missile flight test, with many variations such as the sudden deflection of two controls to give the missile a spiral motion from which various information could be computed. Some of the advances made in the dynamics of torpedoes appear to be transferable to aerodynamic cases.

All in all, there seems to be little question that the Germans fully appreciated the potentialities of dynamics and the utter dependence of future airplane and missile design upon it.

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Letter to the Editor

Dear Sir:

In the excellent summary of the results from the spark photography range in their relation to aerodynamics (Some Ballistic Contributions to Aerodynamics, A. C. Charters, JOURNAL OF THE AERONAUTICAL SCIENCES, Vol. 14, p. 155, March, 1947), Dr. Charters makes reference to some work* by the present writer on head-base interlocking effects on the drag coefficient. The statement is made that this analysis shows the drag to be roughly independent of body length for bodies one to three calibers long. I feel obliged to point out, however, that although the analysis referred to was characterized by a very limited amount of data, the results indicated a systematic dependence of drag coefficient upon body length. That which makes the results worthy of note was the direction of the effect, the projectile with the longer body exhibiting the *smaller* value of the drag coefficient.

The experimental data consist of drag coefficient measures on 19 projectiles with body length ranging from 1.84 to 2.00 in., and on four projectiles with body length 2.79 in., using the experimental values of K_D for the 19 "short" projectiles and smoothing via the "Q" factor described in Charters' paper, K_D values for the Mach Numbers at which the four "long" projectiles were tested and computed. These were compared with the observed results and corrected for yaw on the "long" projectiles.

M	K_D (Short)	K_D (Long)
1.848	0.146	0.143
1.899	0.144	0.141
2.434	0.121	0.120
2.527	0.118	0.115

Obviously, the material is hardly complete enough to warrant quantitative conclusion. Nonetheless, in view of the internal accuracy of the results (about 1 per cent), the effect must be considered real. The models were identical save for body length. The longer bodies cannot fail to have an added contribution to K_D arising from the added frictional drag. (K_D is based on the cross-sectional area.) Thus, the result must be taken to indicate some effect of added body length upon conditions at the base.

A tentative suggestion, but only tentative, arises from reflection upon the properties of a gas expanding around a corner. For the same free-stream pressure, a decrease in the free-stream velocity, holding fast the angle of expansion at the corner, corresponds to an increase in pressure in the stream after the expansion. Thus, for a decrease in the velocity of the gas along the projectile body, accompanying in any way the added body length, we might expect to presage a decrease in drag. This is the effect observed.

RICHARD N. THOMAS
Harvard College Observatory

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Motion of an Unarticulated Helicopter Blade*

JONATHAN WINSON†

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SUMMARY

This paper investigates the bending oscillation perpendicular to the plane of rotation of the unarticulated helicopter blade. The motion is comparable to the flapping of the hinged rotor blade.

Two differential equations of motion are written in terms of two of the natural modes of the blades. The solutions of the equations are a pair of infinite Fourier series in terms of the position of the blade in azimuth. The series are approximated by their constant and first harmonic terms. Six simultaneous linear algebraic equations written in terms of flight and blade parameters give the six Fourier coefficients. A sample problem is carried through, beginning with the physical characteristics of a blade and yielding the bending motion at a specific flight condition.

The method can be expanded both to include a larger number of natural modes and to achieve a closer approximation to the infinite series solutions. There are included a sufficient number of variables to describe blades of current design.

INTRODUCTION

THE FLUCTUATING AERODYNAMIC FORCES that cause flapping in the articulated rotor bend the blades of the unarticulated or rigid rotor in a vertical plane. This pure bending oscillation is treated here. Coupled motions are not considered. They are effectively eliminated by the assumption that the mass center, aerodynamic center, and elastic axis of the blade are at a point at each cross section and that the elastic axis is a straight line.

The method that is presented predicts the motion of the blade under the condition of steady forward flight of the aircraft. It is the direct counterpart of the solution for the flapping motion of the articulated rotor and is the first step in the rational analysis of the rigid rotor, since the flapping equations are the necessary beginning of hinged rotor analysis.

The immediate external effect of the bending of the blade is to cause fluctuating moments on the rotor hub. These moments are the most obvious cause of rigid rotor vibration and a basis for the investigation of such vibration is one of the most important functions of the present analysis. In addition, knowledge of the bending motion yields such information as the fatigue cycle of the blades, the lag angle required in the aircraft control system, and the maximum deflection of the blades.

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* From a thesis written in partial fulfillment of the requirements for the Professional Degree in Aeronautical Engineering at the California Institute of Technology.

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OUTLINE

The work is divided into the following five sections:

I. *The Derivation of the Bending Forces That Exist in Forward Flight Along a Blade.*

The differential force is expressed as a three-harmonic Fourier series in terms of the azimuth position of the blade. The coefficient of each term is a power series in blade radius. The aerodynamic theory is based on work by Glauert.[1]

II. *The Calculation of the Natural Modes and Frequencies of the Blade.*

(a) The method of Myklestad[2] is summarized. By considering the blade made up of a finite number of discrete masses, a tabular method is developed which allows the calculation of the natural modes and frequencies of the blade in its centrifugal field.

(b) A sample blade is considered and the three lowest modes and frequencies are found.

III. *The Derivation and Solutions of the Equations of Motion.*

LaGrange's equation is applied to the system using as generalized coordinates arbitrary functions of time which multiply the two lowest modes. The generalized force is derived in terms of the aerodynamic force. There result two simultaneous second order linear differential equations with variable coefficients. The solution of the equations is a pair of infinite Fourier series in terms of the azimuth angle. The series are approximated by their constant and first harmonic terms. The final solution for the blade motion is given by a set of six simultaneous algebraic equations in terms of the blade and flight parameters. This application of the LaGrangian was suggested by work of M. A. Biot.[3]

IV. *A Numerical Example.*

The calculation of the motion of the sample blade is carried out and plotted at a condition of forward flight at 100 m.p.h.

V. *Discussion.*

A discussion of the assumptions involved and comment on the application of the method are given.

SECTION I.—THE AERODYNAMIC FORCES ON A ROTATING BLADE

Nomenclature

R = radius of the blade

r = coordinate distance along the blade span

y = coordinate distance perpendicular to blade span, defining position of deflected blade with respect to its unbent position

c = chord length at any blade section

c_0 = chord length at the root of any tapered portion of the blade

K_c = coefficient describing the rate of decrease of chord with span for any tapered portion of the blade. For a blade linearly tapered from root to tip: $c = c_0 - K_c(r/R)$

The plane of the rotor disc is the plane perpendicular to rotor shaft passed through the rotor hub.

θ_0 = geometrical blade setting of the blade at its root measured from the line of no lift of the airfoil section

K_θ = coefficient describing the rate of change of blade angle with span in a linearly twisted blade. Thus, $\theta = \theta_0 - K_\theta(r/R)$ indicates linear washout from root to tip

θ = geometrical blade angle at any station along the span, in general a function of r and time

ψ = azimuth position of the blade measured in the direction of rotor rotation. The blade in its downwind position is at zero angle of ψ

u = velocity of flow induced through the rotor disc, assumed constant over the disc

V_r = component of the resultant velocity of the air relative to the blade in a plane perpendicular to the blade span

V_{ry} = component of V_r perpendicular to the blade span and lying in the plane formed by the blade span and the rotor axis of rotation.

V_{rz} = component of V_r which is perpendicular to both V_{ry} and the blade span

dT = thrust component of resultant aerodynamic force on segment of blade dr. dT is parallel to V_{ry}

θ_1, θ_2 = constants that are used to define the cyclic control imposed on the rotor. Arbitrary feathering control produced by a swashplate may be expressed as $\Delta\theta = \theta_1 \sin\psi + \theta_2 \cos\psi$

α = absolute angle of attack of the blade element

ϕ = the induced angle; the angle between V_r and V_{rz}

ω = speed of angular rotation of the rotor

i = angle of attack of the rotor disc; the angle between the flight path and the rotor disc, positive when the disc is tilted forward

ρ = density of the atmosphere

a_∞ = slope of the lift curve for infinite aspect ratio

C_L = coefficient of lift of the blade element

t = time

Thrust Loading of the Blade

Fig. 1 illustrates the velocities and forces at a blade element. The following assumptions are made:

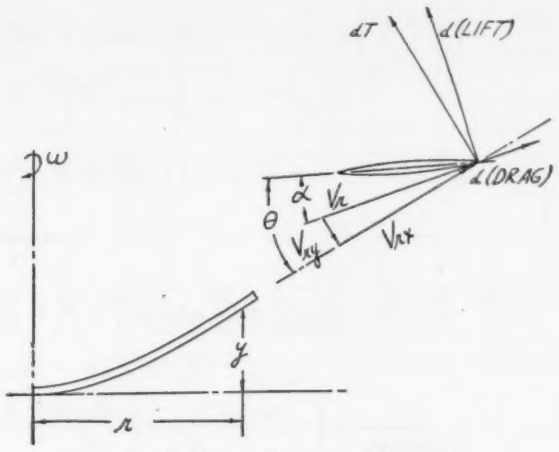

FIG. 1. Velocities and forces at a blade element.

(1) The angle ϕ is small so that $\phi = V_{ry}/V_{rz}$ and $\cos\phi = 1$.

(2) $|V_r| = |V_{rz}|$.

(3) The blade sections operate below the stall.

(4) u is constant over the disc.

(5) The component of blade element drag in the direction of thrust is negligible so that $d(\text{Lift}) = dT$.

(6) $\partial y/\partial r$ is small so that $\sin[\arctan(\partial y/\partial r)] = \partial y/\partial r$.

Then:

$$dT = (\rho/2)C_L V_r^2 c\, dr \tag{1}$$

$$C_L = a_\infty \alpha \tag{2}$$

$$\alpha = \theta - (V_{ry}/V_{rz}) \tag{3}$$

$$\theta = \theta_0 - K_\theta \frac{r}{R} + \theta_1 \sin\psi + \theta_2 \cos\psi \tag{4}$$

$$c = c_0 - K_c(r/R) \tag{5}$$

$$V_{rz} = \omega r + V \cos i \sin\psi \tag{6}$$

$$V_{ry} = V \sin i + u + V \cos i \cos\psi \frac{\partial y}{\partial r} + \frac{\partial y}{\partial t} \tag{7}$$

Let there be defined:

$$\mu = V \cos i/\omega R \tag{8}$$

$$\nu = (V \sin i + u)/\omega R \tag{9}$$

Eq. (2) through Eq. (9) are substituted in Eq. (1). There result three parts to the expression for dT: one independent of blade shape, one dependent on blade shape, and one dependent on the rate of change of the shape with time. Each part is expressed as a Fourier series in ψ. The coefficients of the Fourier series are arranged as power series in r. The final expression for dT is:

$$dT = \omega^2 dr[(A_0 + A_1 \sin\psi + A_2 \cos\psi + A_3 \sin 2\psi + A_4 \cos 2\psi + A_5 \sin 3\psi + A_6 \cos 3\psi) + (\partial y/\partial r)(B_2 \cos\psi + B_3 \sin 2\psi) + (1/\omega)(\partial y/\partial t)(C_0 + C_1 \sin\psi)] \tag{10}$$

where

$$\frac{A_0}{(\rho/2)a_\infty} = \left[\frac{c_0\theta_0\mu^2R^2}{2}\right] + r\left[-\frac{\mu^2R}{2}(c_0K_\theta + \theta_0K_c) + c_0R(\theta_1\mu - \nu)\right] + r^2\left(c_0\theta_0 + \frac{K_cK_\theta\mu^2}{2} - K_c\theta_1\mu + K_c\nu\right) + r^3\left(-\frac{c_0K_\theta}{R} - \frac{K_c\theta_0}{R}\right) + r^4\left(\frac{K_cK_\theta}{R^2}\right) \quad (11)$$

$$\frac{A_1}{(\rho/2)a_\infty} = \left(\frac{3}{4}c_0\theta_1\mu^2R^2 - c_0\mu\nu R^2\right) + r\left(-\frac{3}{4}K_c\theta_1\mu^2R + 2c_0\theta_0\mu R + K_c\mu\nu R\right) + r^2(c_0\theta_1 - 2c_0K_\theta\mu - 2K_c\theta_0\mu) + r^3\left(-\frac{K_c\theta_1}{R} + \frac{2K_cK_\theta\mu}{R}\right) \quad (12)$$

$$\frac{A_2}{(\rho/2)a_\infty} = \left(\frac{1}{4}c_0\theta_2\mu^2R^2\right) + r\left(-\frac{1}{4}K_c\theta_2\mu^2R\right) + r^2\left(c_0\theta_2\right) + r^3\left(-\frac{K_c\theta_2}{R}\right) \quad (13)$$

$$A_3/(\rho/2)a\infty = r(c_0\theta_2\mu R) + r^2(-K_c\theta_2\mu) \quad (14)$$

$$\frac{A_4}{(\rho/2)a_\infty} = \left(-\frac{c_0\theta_0\mu^2R^2}{2}\right) + r\left(\frac{c_0K_\theta\mu^2R}{2} + \frac{K_c\theta_0\mu^2R}{2} - c_0\theta_1\mu R\right) + r^2\left(-\frac{K_cK_\theta\mu^2}{2} + K_c\theta_1\mu\right) \quad (15)$$

$$A_5/(\rho/2)a_\infty = (-{}^1/_4c_0\theta_1\mu^2R^2) + r({}^1/_4K_c\theta_1\mu^2R) \quad (16)$$

$$A_6/(\rho/2)a_\infty = (-{}^1/_4c_0\theta_2\mu^2R^2) + r({}^1/_4K_c\theta_2\mu^2R) \quad (17)$$

$$B_2/(\rho/2)a_\infty = r(-c_0\mu R) + r^2(K_c\mu) \quad (18)$$

$$B_3/(\rho/2)a_\infty = (-c_0\mu^2R^2/2) + r(K_c\mu^2R/2) \quad (19)$$

$$C_0/(\rho/2)a_\infty = r(-c_0) + r^2(K_c/R) \quad (20)$$

$$C_1/(\rho/2)a_\infty = (-c_0\mu R) + r(K_c\mu) \quad (21)$$

Section IIA.—The Natural Modes and Frequencies of the Rotating Blade

The Myklestad method for the calculation of the natural modes and frequencies of a rotating blade may be found in reference 2. A word summary of the method is given here.

The theory of normal coordinates indicates that, at a natural frequency, the rotating cantilever beam considered will vibrate in such a manner that the following conditions are satisfied:

(1) The transverse motion of all points on the beam will be harmonic and in phase.

(2) The motion will exist without application of external forcing functions. The external forces acting on a point mass will be only those of inertia and centrifugal force.

(3) The motion will be consistent with the elastic characteristics of the beam.

(4) The boundary conditions of the system will be satisfied. They are: (a) deflection at the root = 0 and (b) slope at the root = 0.

By a proper mathematical statement of the four conditions, it is possible to find both the frequencies at which this motion will exist and the mode of motion at each frequency.

More specifically, the system is assumed to consist of a finite number of discrete masses connected by a massless elastic beam. The four points mentioned above are imposed with the exception of boundary condition 4b. This boundary condition is made a function of an assumed natural frequency of the blade. When the assumed natural frequency is the true natural frequency, the boundary condition is satisfied; that is, the slope at the root of the blade is zero. Thus, a plot of assumed natural frequency against calculated value of the slope of the blade at its root yields as many of the natural frequencies as are desired. The mode associated with each frequency is found with little additional calculation.

Section IIB.—Natural Frequencies and Modes of the Sample Blade

The sample blade considered is shown in Fig. 2. Its main structural member is a step-tapered hollow steel spar. So little stiffness in vertical bending is added by

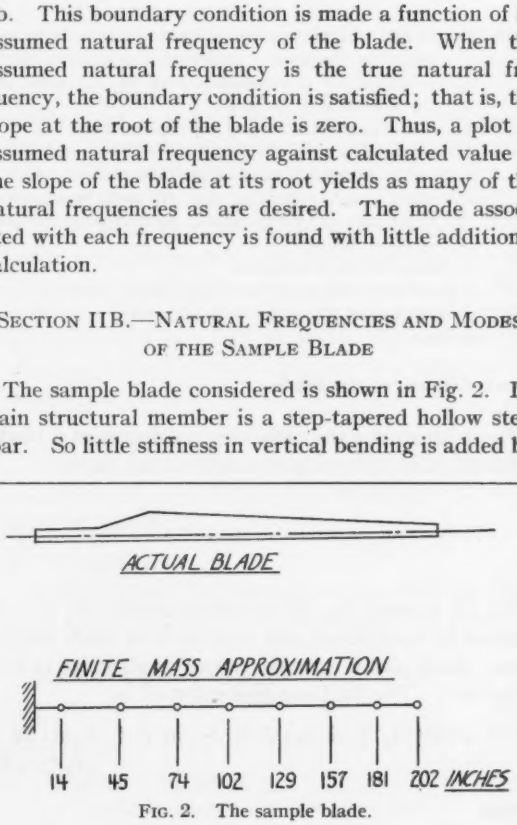

Fig. 2. The sample blade.

TABLE 1
Physical Characteristics of the Sample Blade

Station, In.	Spar Size, In.	Weight, Lbs.
0– 12	3.50 × 0.263	9.09
12– 30	3.25 × 0.224	10.86
30– 45	3.00 × 0.200	9.08
45– 60	2.75 × 0.183	9.21
60– 74	2.50 × 0.170	8.02
74– 90	2.25 × 0.148	7.80
90–102	2.00 × 0.131	5.02
102–114	1.75 × 0.117	4.30
114–126	1.50 × 0.109	3.79
126–145	1.25 × 0.104	5.17
145–169	1.00 × 0.104	5.66
169–193	0.875 × 0.102	4.90
193–210	0.750 × 0.102	3.05
		85.95 lbs.

TABLE 2
Finite Mass Approximation of the Blade

Station, In.	Concentrated Mass, Lbs.Sec.²/In.
14	0.0517
45	0.0474
74	0.0410
102	0.0241
129	0.0232
157	0.0147
181	0.0127
202	0.00790

the covering that the spar is considered to be the only elastic member present. The weights of the covering and ribs are taken into account.

The essential characteristics of the blade are given in Table 1. Spar size is written as diameter × wall thickness. Table 2 lists the system of masses chosen to represent the blade. Fig. 3 is the plot of assumed frequency versus slope at the root of the blade which was used to find the three lowest natural frequencies. Fig. 4 is a plot of the modes associated with these frequencies.

SECTION III.—MOTION OF THE BLADE

The motion of the blade under the applied aerodynamic forces will now be found. In this section the system is considered continuous rather than made up of a finite number of discrete masses.

The motion, in general, will consist of some combination of the natural modes. Only the effects of the lowest two natural modes are considered here. Section V discusses this point.

The normal modes (normalized by taking unit deflection at the blade tip) may be described by power series:

$$y = y_1(r) \tag{22}$$

and

$$y = y_2(r) \tag{23}$$

The complete motion may be expressed as:

$$y = q_1 y_1 + q_2 y_2 \tag{24}$$

where

$$q_1 = q_1(t) \tag{25}$$

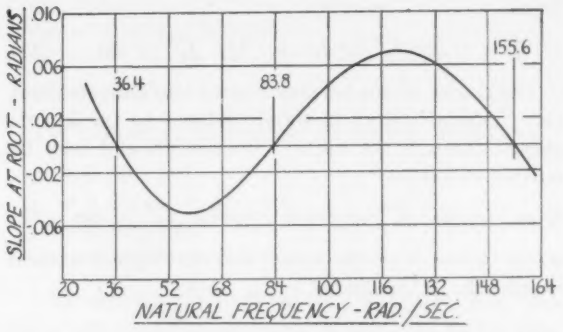

FIG. 3. Assumed natural frequency versus slope at the root of the blade.

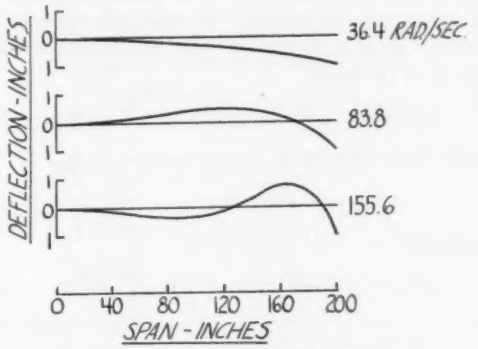

FIG. 4. The three lowest natural modes.

$$q_2 = q_2(t) \tag{26}$$

q_1 and q_2 are taken as the generalized coordinates in an application of the LaGrange equation. The generalized forces are found from the expressions for the aerodynamic force.

LaGrange's Equation:

$$(d/dT)(\partial T/\partial \dot{q}_i) + (\partial U/\partial q_i) = Q_i \tag{27}$$

where

T = the kinetic energy of the system
U = the potential energy of the system
Q_i = the generalized force, as defined later
$\dot{q}_i = dq_i/dt$

The Kinetic and Potential Energies:

$$T = {}^1/_2 \int_0^R \dot{y}^2 \, dm \tag{28}$$

where dm is the differential mass. Substituting Eq. (24) in Eq. (28) and squaring,

$$T = {}^1/_2 \int_0^R \dot{q}_1^2 y_1^2 \, dm + {}^1/_2 \int_0^R \dot{q}_2^2 y_2^2 \, dm + \int_0^R \dot{q}_1 \dot{q}_2 y_1 y_2 \, dm \tag{29}$$

Because of the orthogonality of the principal modes,

$$\int_0^R y_1 y_2 \, dm = 0 \tag{30}$$

Thus,

$$T = {}^1/_2\dot{q}_1^2 \int_0^R y_1^2\, dm + {}^1/_2\dot{q}_2^2 \int_0^R y_2^2\, dm \qquad (31)$$

The theory of normal coordinates also indicates that the potential energy is simply related to the kinetic energy through the natural frequencies and may be written as follows:[6]

$$U = {}^1/_2 q_1^2\Omega_1^2 \int_0^R y_1^2\, dm + {}^1/_2 q_2^2\Omega_2^2 \int_0^R y_2^2\, dm \qquad (32)$$

where Ω_1 and Ω_2 are the lowest and next highest natural frequencies of the blade.

The Generalized Force:

Q_i is defined as $\delta W/\delta q_i$, where δW is the work done by the external forces on the system during a small change in the coordinate q_i. The small change in coordinate is indicated by δq_i. In this case the only external load is that of the air. Its magnitude is dT in length dr of the span.

$$\delta W = \int dT\, \delta y \qquad (33)$$

over the blade, where δy occurs because of δq_i.

$$\delta y = \delta q_i y_i \qquad (34)$$

Substituting Eq. (34) in Eq. (33),

$$\delta W = \int_0^R dT\, y_i \delta q_i = \delta q_i \int_0^R dT\, y_i \qquad (35)$$

Thus,

$$Q_i = \int_0^R dT\, y_i \qquad (36)$$

The equation of motion for the coordinate q_1 is now written in the following manner:

(1) Eq. (24) is substituted in Eq. (10).

(2) The resulting expression for dT is substituted in Eq. (36), i being set equal to 1.

(3) Eqs. (31) and (32) and the expression for Q_1 found in step 2 above are substituted in Eq. (27), the LaGrange equation.

(4) The symbolism $y_i' = dy_i/dr$ is introduced and the following constants are defined:

$$\begin{aligned}
\lambda_1 &= \int_0^R y_1^2\, dm \\
\lambda_3 &= \int_0^R y_1 A_0\, dr \\
\lambda_4 &= \int_0^R y_1 A_1\, dr \\
\lambda_5 &= \int_0^R y_1 A_2\, dr \\
\lambda_6 &= \int_0^R y_1 A_3\, dr \\
\lambda_7 &= \int_0^R y_1 A_4\, dr \\
\lambda_8 &= \int_0^R y_1 A_5\, dr \\
\lambda_9 &= \int_0^R y_1 A_6\, dr \\
\lambda_{10} &= \int_0^R y_1 y_1' B_2\, dr \\
\lambda_{11} &= \int_0^R y_1 y_1' B_3\, dr \\
\lambda_{12} &= \int_0^R y_1 y_2' B_2\, dr \\
\lambda_{13} &= \int_0^R y_1 y_2' B_3\, dr \\
\lambda_{14} &= \int_0^R y_1^2 C_0\, dr \\
\lambda_{15} &= \int_0^R y_1^2 C_1\, dr \\
\lambda_{16} &= \int_0^R y_1 y_2 C_0\, dr \\
\lambda_{17} &= \int_0^R y_1 y_2 C_1\, dr
\end{aligned}$$

A change of variable is made. Recognizing that $\psi = \omega t$, the final equation is written in terms of ψ. In terms of this variable the differential equation of motion for the coordinate q_1 becomes:

$$\begin{aligned}
&\lambda_1 \frac{d^2q_1}{d\psi^2} + \frac{dq_1}{d\psi}(-\lambda_{14} - \lambda_{15}\sin\psi) + \\
&q_1\left(\frac{\Omega_1^2}{\omega^2}\lambda_1 - \lambda_{10}\cos\psi - \lambda_{11}\sin 2\psi\right) + \frac{dq_2}{d\psi} \times \\
&(-\lambda_{16} - \lambda_{17}\sin\psi) + q_2(-\lambda_{12}\cos\psi - \lambda_{13}\sin 2\psi) = \\
&\lambda_3 + \lambda_4\sin\psi + \lambda_5\cos\psi + \lambda_6\sin 2\psi + \\
&\lambda_7\cos 2\psi + \lambda_8\sin 3\psi + \lambda_9\cos 3\psi \qquad (37)
\end{aligned}$$

A similar set of substitutions, in this case with i being set equal to 2, leads to the equation for the variable q_2. The set of constants are:

$$\begin{aligned}
\lambda_2 &= \int_0^R y_2^2\, dm \\
\lambda_3' &= \int_0^R y_2 A_0\, dr \\
\lambda_4' &= \int_0^R y_2 A_1\, dr \\
\lambda_5' &= \int_0^R y_2 A_2\, dr \\
\lambda_6' &= \int_0^R y_2 A_3\, dr \\
\lambda_7' &= \int_0^R y_2 A_4\, dr \\
\lambda_8' &= \int_0^R y_2 A_5\, dr \\
\lambda_9' &= \int_0^R y_2 A_6\, dr \\
\lambda_{10}' &= \int_0^R y_2 y_1' B_2\, dr \\
\lambda_{11}' &= \int_0^R y_2 y_1' B_3\, dr \\
\lambda_{12}' &= \int_0^R y_2 y_2' B_2\, dr \\
\lambda_{13}' &= \int_0^R y_2 y_2' B_3\, dr \\
\lambda_{14}' &= \int_0^R y_2 y_1 C_0\, dr \\
\lambda_{15}' &= \int_0^R y_2 y_1 C_1\, dr \\
\lambda_{16}' &= \int_0^R y_2^2 C_0\, dr \\
\lambda_{17}' &= \int_0^R y_2^2 C_1\, dr
\end{aligned}$$

The equation is:

$$\begin{aligned}
&\lambda_2 \frac{d^2q_2}{d\psi^2} + \frac{dq_2}{d\psi}(-\lambda_{16}' - \lambda_{17}'\sin\psi) + \\
&q_2\left(\frac{\Omega_2^2}{\omega^2}\lambda_2 - \lambda_{12}'\cos\psi - \lambda_{13}'\sin 2\psi\right) + \frac{dq_1}{d\psi} \times \\
&(-\lambda_{14}' - \lambda_{15}'\sin\psi) + q_1(-\lambda_{10}'\cos\psi - \\
&\lambda_{11}'\sin 2\psi) = \lambda_3' + \lambda_4'\sin\psi + \lambda_5'\cos\psi + \\
&\lambda_6'\sin 2\psi + \lambda_7'\cos 2\psi + \lambda_8'\sin 3\psi + \\
&\lambda_9'\cos 3\psi \qquad (38)
\end{aligned}$$

Solution of the Equations:

The two infinite Fourier series, which are the solutions of Eqs. (37) and (38), are approximated by their constant and first harmonic terms. Thus,

$$q_1 = a_0 + a_1\sin\psi + a_2\cos\psi \qquad (39)$$

$$q_2 = b_0 + b_1\sin\psi + b_2\cos\psi \qquad (40)$$

Substitution of Eqs. (39) and (40) in Eqs. (37) and (38) and harmonic analysis leads to the following six algebraic equations for the six Fourier coefficients:

$$\left(\frac{\Omega_1^2}{\omega^2}\right)\lambda_1 a_0 + \frac{\lambda_{15} - \lambda_{10}}{2} a_2 + (\lambda_{17} - \lambda_{12})b_2 = \lambda_3 \quad (41)$$

$$\frac{\lambda_{15}' - \lambda_{10}'}{2} a_2 + \left(\frac{\Omega_2^2}{\omega^2}\right)\lambda_2 b_0 + (\lambda_{17}' - \lambda_{12}')b_2 = \lambda_3' \quad (42)$$

$$\lambda_1\left(\frac{\Omega_1^2}{\omega^2} - 1\right)a_1 + \left(\lambda_{14} - \frac{\lambda_{11}}{2}\right)a_2 + \left(\lambda_{16} - \frac{\lambda_{13}}{2}\right)b_2 = \lambda_4 \quad (43)$$

$$\left(\lambda_{14}' - \frac{\lambda_{11}'}{2}\right)a_2 + \lambda_2\left(\frac{\Omega_2^2}{\omega^2} - 1\right)b_1 + \left(\lambda_{16}' - \frac{\lambda_{13}'}{2}\right)b_2 = \lambda_4' \quad (44)$$

$$\lambda_{10}a_0 + \left(\lambda_{14} + \frac{\lambda_{11}}{2}\right)a_1 - \lambda_1\left(\frac{\Omega_1^2}{\omega^2} - 1\right)a_2 + \lambda_{12}b_0 + \left(\lambda_{16} + \frac{\lambda_{13}}{2}\right)b_1 = -\lambda_5 \quad (45)$$

$$\lambda_{10}'a_0 + \left(\lambda_{14}' + \frac{\lambda_{11}'}{2}\right)a_1 + \lambda_{12}'b_0 + \left(\lambda_{16}' + \frac{\lambda_{13}'}{2}\right)b_1 - \lambda_2\left(\frac{\Omega_2^2}{\omega^2} - 1\right)b_2 = -\lambda_5' \quad (46)$$

The solution of these six equations gives the motion of the blade at a specific flight condition within the approximations made.

Section IV.—Numerical Example

The motion of the sample blade at a given flight condition will now be found. The following quantities must be evaluated:

$y_1(r), y_2(r)$; y_1', y_2'; Ω_1, Ω_2; $A_0, A_1, A_2, B_2, B_3, C_0, C_1$; $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}, \lambda_{14}, \lambda_{15}, \lambda_{16}, \lambda_{17}, \lambda_3', \lambda_4', \lambda_5', \lambda_{10}', \lambda_{11}', \lambda_{12}', \lambda_{13}', \lambda_{14}', \lambda_{15}', \lambda_{16}', \lambda_{17}'$

Quantities y_1, y_2, y_1', y_2':

The following series are adequate for the representation of the two modes:

$$y_1 = 0.00354r^2$$
$$y_2 = -0.009398r^2 + 0.00004584r^4$$

where r is in feet. Then,

$$y_1' = 0.00708r$$
$$y_2' = -0.018796r + 0.00018336r^3$$

Physical Constants:

$c_0 = 1.593$ ft.
$K_c = 1.039$ ft.
$K_\theta = 0$
$R = 17.5$ ft.
$\omega = 26$ rad. per sec.
$\rho = 0.002378$ slugs per cu.ft.
$a_\infty = 5.3$
$dm = (0.258 - 0.0120r)dr$

Flight Condition:

The following aerodynamic parameters are taken:

$\mu = 0.304$
$\nu = 0.0853$
$\theta_0 = 0.2269$ rad.
$\theta_1 = \theta_2 = 0$

This corresponds to flight at 100 m.p.h. of a ship using two two-bladed rotors of the type being considered. The weight of the ship is approximately 3,000 lbs.

Aerodynamic Quantities:

From Eqs. (11) through (21),
$A_0 = 0.03223 - 0.01619r + 0.002837r^2 - 0.00008489r^3$
$A_1 = -0.07972 + 0.02721r - 0.0009031r^2$
$A_2 = 0$
$B_2 = -0.05341r + 0.001991r^2$
$B_3 = -0.1421 + 0.005296r$
$C_0 = -0.01004r + 0.0003741r^2$
$C_1 = -0.05341 + 0.001990r$

Integrated Constants:

$\lambda_1 = 0.3414$
$\lambda_2 = 0.2914$
$\lambda_3 = 0.7181$
$\lambda_4 = 0.7052$
$\lambda_5 = 0$
$\lambda_{10} = -0.2005$
$\lambda_{11} = -0.03994$
$\lambda_{12} = -0.5364$
$\lambda_{13} = -0.08872$
$\lambda_{14} = -0.2657$
$\lambda_{15} = -0.1003$
$\lambda_{16} = -0.04737$
$\lambda_{17} = -0.00099$
$\lambda_3' = -0.08594$
$\lambda_4' = -0.07300$
$\lambda_5' = 0$
$\lambda_{10}' = -0.00204$
$\lambda_{11}' = 0.00866$
$\lambda_{12}' = -0.3281$
$\lambda_{13}' = -0.05793$
$\lambda_{14}' = -0.04737$
$\lambda_{15}' = -0.00099$
$\lambda_{16}' = -0.1886$
$\lambda_{17}' = -0.08088$
$(\Omega_1/\omega)^2 = 1.9600$
$(\Omega_2/\omega)^2 = 10.388$

Substituting in Eqs. (41) through (46) and solving, there is obtained:

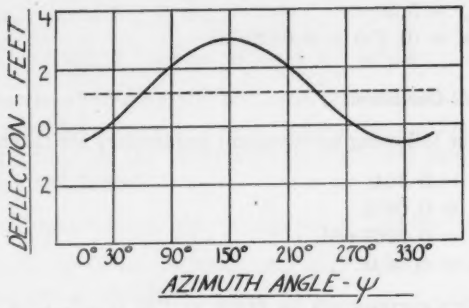

FIG. 5. Deflection at the tip of the blade; flight at 100 m.p.h.

$$a_0 = +1.20$$
$$a_1 = +1.00$$
$$a_2 = -1.54$$
$$b_0 = -0.03$$
$$b_1 = -0.06$$
$$b_2 = -0.01$$

The motion is then described by

$$y = (1.20 + 1.00 \sin \psi - 1.54 \cos \psi)(0.00354r^2) + (-0.03 - 0.06 \sin \psi - 0.01 \cos \psi)(-0.009398r^2 + 0.00004584r^4)$$

where r is in feet.

The motion of a point at the tip of the blade is plotted in Fig. 5.

SECTION V.—DISCUSSION

Aerodynamics

The aerodynamic derivation is based on steady flow wing theory. Air forces are determined by instantaneous values of velocity and angle of attack at each blade element. However, the following facts are known to be incompatible with the assumption of steady flow:

(1) The blade element oscillates vertically with an amplitude that is large as compared to the chord.

(2) The blade element changes its pitch periodically.

(3) The velocity at the blade element varies greatly.

(4) The blade operates in a wake produced by itself, other blades of the rotor, and other rotor(s).

Nevertheless, the Glauert steady flow theory[1] has been widely applied in helicopter work and has given results that have been found reasonable. Specifically, the calculation of the flapping motion of hinged blades has been carried out using this theory and has proved to be in satisfactory agreement with experiment. A more exact theory has not yet been developed. Points 1 and 2 have been considered for the case of small oscillations by von Kármán and Sears[4] and point 3 by Isaacs,[5] but no complete evaluation of the Glauert theory can be made on the basis of these. The one approximate calculation that can be made does not indicate a large error.*

The Differential Equations

The differential equations are not exact in that the modes assumed to make up the motion and their natural frequencies are calculated for the undamped, conservative system. This is the most compromising of the assumptions made and cannot be rigorously defended without an exact solution of the problem. It has some precedent in flutter work and in the assumptions made in lesser degree-of-freedom systems (that a reasonable amount of damping does not change the undamped natural frequency), but the assumption is intuitive. Structural damping is neglected as unimportant as compared to the aerodynamic damping.

Solution of the Differential Equations

The solutions of the differential equations are a pair of infinite Fourier series in ψ. The justification for approximating the series by constant and first harmonic terms is based on comparison with the flapping equation for hinged blades, a degenerate case of the equations derived here. The forcing function in both systems is characterized by strong first harmonic terms with higher harmonics of rapidly diminishing magnitude.[7] An exact solution of the flapping case[8] shows a corresponding, characteristic, strong first harmonic response with rather small higher harmonic motions. The inclusion of two modes in the present analysis allows any higher harmonic to be considered whose proximity to resonance emphasizes its importance. There is no theoretical barrier to carrying the series solutions out to higher harmonics.

* Assuming small vertical oscillations, the effects of deviations 1 and 2 are found to be small. The reduced frequency of the system as defined in the von Kármán paper is of the order of 0.04.

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In-Flight Icing of Highly Electrified Aircraft

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ABSTRACT

The suggestion that a highly electrified airplane will not ice up in normal icing conditions is examined. Tests with an artificial charger on an aircraft in flight and on electrified probes mounted on an experimental airplane have shown that the electrification has no measurable effect on aircraft icing.

INTRODUCTION

THE SERIOUS HAZARD resulting from the formation of ice on flying aircraft has engaged the attention of aeronautical engineers and scientists for many years, and various suggestions have been made for its elimination. It has long been known that the water particles in the atmosphere carry appreciable electric charges. A short calculation using available data[1] shows that the electrical charges on the droplets are sufficiently large so that one might reasonably expect the forces on water droplets, when exposed to actually observed electric fields, to affect appreciably the deposition of ice-forming particles on aircraft in flight. This possibility has been pointed out by several investigators. H. B. Barakan, of the Leningrad Institute of Meteorology, has published a paper[2] entitled "One of the Possible Causes of Airplane Icing" in which he suggests that the electrification of aircraft plays an important part in the icing of aircraft. More recently, a U.S. Patent No. 2,347,114 has been issued on a "Method and Means for Preventing Ice Formation on Surfaces" in which definite claims are made as to the influence of a large free charge carried by the airplane on the icing of the frontal surfaces and which particularly states that a suitable electrostatic charge on the surface of the airplane will prevent icing.

RESEARCH

The possible importance of any method for suppressing icing of aircraft and the fact that the suggestions appeared to have some quantitative foundation encouraged the research team to explore the problem. Experience and flight measurements have shown that the electrical state of water particles sprayed into a wind tunnel is in no way related to the charge on natural precipitation droplets. It was a foregone conclusion, therefore, that experiments looking toward a complete answer to this important problem and carried on in a wind tunnel would be subject to failure or to gross misinterpretation, because the water droplets in a wind tunnel would not be sufficiently aged to pick up equilibrium electrical charges such as are encountered in actual ice-forming clouds.

In view of these facts and because there were available at Minneapolis an airplane and facilities that could be used without diverting them from their principal duty, it was decided to test the idea under actual flying conditions. By special apparatus, the B-25 assigned to the project could be charged artificially in flight up to 250,000 volts and of either sign.[3] Two preliminary experimental flights using this artificial charging were unsuccessful because the research team could not be sure of finding identical icing conditions on two successive runs. In order to avoid this difficulty, another simple experimental arrangement was adopted and led to a clear-cut result. In this experiment three rods $^3/_8$ in. in diameter and effective length of $4^1/_2$ in. were turned out of brass and carefully smoothed. These three identical rods were mounted on the top of the B-25 just behind the pilot in the Plexiglas window and about 4 in. in front of the aluminum skin. The three rods were provided with bosses and mounted symmetrically with respect to the axis of the ship and spaced from each other by $1^1/_2$ in. The three rods are shown in Fig. 1 (the streamline body immediately behind the rods has no bearing on this experiment). A high voltage source was arranged so that any voltage from 0 to $\pm 10{,}000$ volts could be placed on the starboard outside rod, while, simultaneously, a voltage of from 0 to $-10{,}000$ volts could be placed on the port outside rod. The middle rod was grounded to the frame of the airplane and had no voltage on it. In general, the rods were operated in such a manner that positive potential of 8,000 volts was placed on the starboard rod while $-8{,}000$ volts was placed on the port rod.

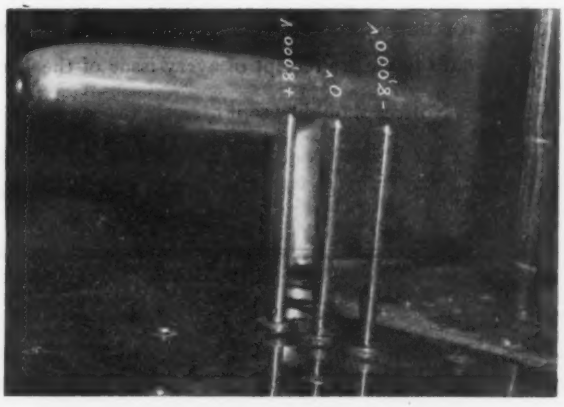

FIG. 1. Electrified probes mounted on B-25.

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The airplane was flown through icing conditions twice with no voltage applied to any of the rods, and these blank runs demonstrated that the position of the rods was such that they acquired ice equally, as determined by observation and measurement.†

With voltages applied to the rods, the ship was flown to the top of the stratus cloud deck above which existed a slight temperature inversion. The electrodes had accumulated some ice during the climb, but this quickly melted clean in the warm air on top of the clouds. The top of the cloud deck was 3,300 ft. above mean sea level, and the temperature on top was 4°C. The high voltage was applied to the electrodes and the airplane dropped to 2,500 ft. into an icing condition that could be described as a moderate glaze in an ambient temperature of −3°C. The airplane remained in this cloud deck for 12 min., and a substantial coating of ice on the electrodes was accumulated. It is estimated that the thickness of the ice layer on the leading edge of the rods was $^3/_4$ in. As far as could be observed no difference in the amount or distribution of ice on any of the three electrodes could be detected. Throughout the run the electric field on the top of the fuselage of the B-25 varied between 50 and 100 volts per cm. (a small value).

Because it is recognized that the electrical effects of icing may depend critically upon meterological conditions, one other careful test was made and other casual runs have been made under different conditions. The second careful run, made on the succeeding day, was much like the first one except the rate of icing was very much lower and the artificial charger on the B-25 was operated in such a way that the airplane itself had no resultant free electrical charge. Again, the ice accretion was identical on all electrodes as far as could be determined. The icing under this condition is shown in Fig. 1. This photograph was taken almost immediately upon the return of the airplane to the hangar.

† The author acknowledges the helpful cooperation of John Randolph Galt, Chief Pilot assigned to the Precipitation Static Project, in running these series of tests.

Conclusions

Although we would not like to claim that a thorough investigation of the matter has been made, it is thought that our experiments demonstrate rather clearly that electrical effects are unimportant in determining the deposition of ice on aircraft. The electrical fields existing in the vicinity of the rod electrodes are approximately as large as any electric field likely to exist on an aircraft even in dry snow conditions. Because of this fact and because ice invariably forms on the leading edge of aircraft, our experiments seem to show that the artificial electrical charging of an aircraft to suppress icing is of no practical interest.

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The Ring Airfoil in Nonaxial Flow

HERBERT S. RIBNER*

National Advisory Committee for Aeronautics

SUMMARY

The ring airfoil with axis inclined to the stream is investigated by a lifting-line theory. The stream tube that threads the ring is found to deflect like a rigid cylinder. The lift is twice the lift of a flat elliptic wing that spans a diameter and has a quarter the area, and the downwash and ratio of lift to induced drag are the same for both.

INTRODUCTION

THE LIFTING-SURFACE THEORY of the ring airfoil in axisymmetric flow has been treated in references 1 and 2. The pressure distribution is correspondingly axisymmetric, and there is no net lift. The ring airfoil with axis inclined to the stream does not appear to have been investigated. Such an airfoil experiences a net lift, and this lift is calculable by a simple lifting-line theory. The approximation is of the same order as in the lifting-line theory of the flat wing. The theory is presented herein.

SYMBOLS

V = stream velocity
ρ = mass density of fluid
R = radius of ring airfoil and of cylindrical wake
c = chord of ring airfoil
y = lateral distance from ring center
θ = arc cos y/R
α = angle of attack
α_i = induced angle of attack
ϵ = angle of downwash
Γ = circulation
L = lift
C_L' = lift coefficient $[L/(1/2)\rho V^2 S']$
S' = area of reference wing $[(\pi/2)\ Rc]$
A = aspect ratio
A' = aspect ratio of reference wing $(8R/\pi c)$

THEORY

Assume, subject to later proof, that the inclined ring airfoil deflects as a rigid body the stream tube that flows through it. Let the angle of deflection (downwash at infinity) be 2ϵ. For small inclination of the ring airfoil, this stream tube is essentially a circular cylinder. The downward deflection implies a distribution of axial vorticity along its boundary of strength:

$$d\Gamma/Rd\theta = 4V\epsilon \cos\theta \qquad (1)$$

where V is the stream velocity and θ the angle with the horizontal. (See Fig. 1.)

The flow velocity induced radially inward at points on the boundary far back is $2V\epsilon \sin\theta$. The radial induced velocity at the end of the cylinder that meets the ring airfoil is just half this. Thus, the induced angle of attack is

$$\alpha_i = \epsilon \sin\theta$$

If the angle of attack of the axis is α, the geometric angle of attack of points along the ring airfoil is $\alpha \sin\theta$. The effective angle of attack is therefore $(\alpha - \epsilon)\sin\theta$.

According to the basic assumption of lifting-line theory,

$$\Gamma = \pi Vc(\alpha - \alpha_i)$$

where c is the section chord and the two-dimensional lift-curve slope is taken as 2π. Therefore,

$$\Gamma = \pi Vc(\alpha - \epsilon)\sin\theta \qquad (2)$$

The strength of the trailing vorticity is thus

$$d\Gamma/Rd\theta = (\pi Vc/R)(\alpha - \epsilon)\cos\theta \qquad (3)$$

Now Eq. (3) shows the cosine distribution of vorticity characteristic of a rigidly deflected cylinder of air. This proves the original assumption. The identification of Eqs. (1) and (3) now leads to the determination of ϵ:

$$4V\epsilon \cos\theta = (\pi Vc/R)(\alpha - \epsilon)\cos\theta$$

$$\epsilon = \alpha/[1 + (4R/\pi c)] \qquad (4)$$

Returning to Eq. (2), the substitution of the value of ϵ from Eq. (4) gives

$$\Gamma = \frac{4R/\pi c}{1 + (4R/\pi c)}\ \pi Vc\ \alpha \sin\theta$$

The outward normal force per unit arc is $\rho V\Gamma$. This is

$$\frac{dN}{Rd\theta} = \frac{4R/\pi c}{1 + (4R/\pi c)}\ \rho V^2 \pi c\alpha \sin\theta$$

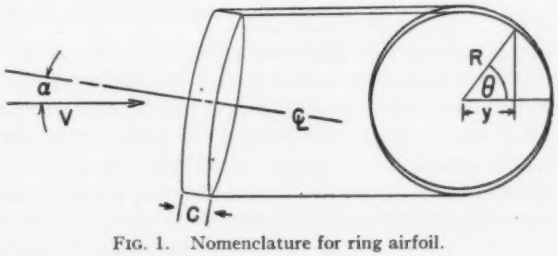

FIG. 1. Nomenclature for ring airfoil.

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The liftwise component of this normal force, expressed per unit span, is the same. Adding contributions from top to bottom introduces a factor of two:

$$\frac{dL}{dy} = 2\frac{4R/\pi c}{1 + (4R/\pi c)}\rho V^2 \pi c\alpha \sin\theta$$

The lift is seen to be distributed elliptically, and by integration the total value is

$$L = \frac{4R/\pi c}{1 + (4R/\pi c)}\rho V^2 \pi^2 Rc\alpha \qquad (5)$$

It will be convenient to introduce a reference elliptic flat plate of span R and area $S' = (\pi/2)Rc$ so that the aspect ratio is $A' = 8R/\pi c$. Substituting in Eq. (5) gives

$$C_L' = L/(1/2)\rho V^2 S' = 4\pi[A'/(A' + 2)]\alpha \qquad (6)$$

Eq. (6) gives precisely twice the lifting-line result for the lift of the flat reference wing. Thus, the lift of the ring airfoil is twice the lift of an elliptic flat plate that spans a diameter and has a quarter the area. Also, the angle of downwash and the ratio of lift to induced drag are the same for the two airfoils, as can be seen either from Eq. (4) or from considerations of momentum.

Eq. (5) or (6) for the lift applies to ring airfoils with diameter large relative to the chord. It may be expected to break down at very low aspect ratios $A' = 8R/\pi c$ just as the corresponding equation for flat elliptic wings breaks down. At these very low aspect ratios the wake would align itself with the axis of the ring, just as it does with the plane of a flat airfoil. The lift of both would then be independent of aspect ratio and would be determined only by α and the span (or α and the radius) (see references 3 and 4). From considerations of momentum the lift of a very low aspect ratio ring should be

$$L = 2\rho\pi R^2 V^2\alpha$$

In Eq. (5) if c is allowed to become very large relative to R,

$$L = 4\rho\pi R^2 V^2\alpha$$

or twice the preceding value. The lift of a very low aspect ratio flat plate of semispan $= R$ should be

$$L = \rho\pi R^2 V^2\alpha$$

The lifting-line theory for the elliptic flat plate gives

$$L = [(1/2)(\rho V^2 S')](2\pi\alpha)[A'/(A' + 2)]$$

and for very low aspect ratio ($A' \to 0$) this becomes

$$L = \rho V^2 \pi\alpha[S'(4R^2/S')/2]$$
$$= 2\rho\pi R^2 V^2\alpha$$

again twice the correct value. Thus, even at very low aspect ratios, when Eqs. (5) and (6) are invalid, the lift of the ring airfoil is still twice the lift of an elliptic flat plate that spans a diameter and has a quarter the area.

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Face Buckling and Core Strength Requirements in Sandwich Construction*

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SUMMARY

Relations between the wrinkling stress of the facings of sandwich materials and the physical properties of the core are studied. The effect of initial waviness on the wrinkling stress and the strength requirements of the core are treated by the introduction of a numerical coefficient K. Charts for estimating wrinkling stresses are presented.

INTRODUCTION

THERE HAS BEEN considerable interest during the past few years in the possible use of sandwich materials for the primary structures of a modern airplane. Such sandwich materials consist of two thin layers of high-strength facings bonded to a thick layer of low-density core. The core, which provides shear connection between the facings, increases the effective moment of inertia of the facing cross section and thus makes possible a considerable improvement in the stability of structural members. Suitable core materials can be found to perform the additional function of stabilizing the facings so that they will not wrinkle until a high state of stress in the material is reached. It is apparent that the core plays an important role in sandwich materials, and it is desirable to know the strength and stiffness requirements for an efficient core material.

The object of this paper is to derive certain theoretical relationships between the wrinkling stress of the facings and the physical properties of the core in sandwich construction so that a proper core material may be selected by means of these criteria. Surface waviness is incorporated in the present analysis, since it has been found that this waviness may have a detrimental effect on the wrinkling stress. Sandwiches under uniform simple compression are treated first to establish the fundamental relationships. These are later extended to include the case of sandwiches under biaxial stresses.

GENERAL DISCUSSION

The wrinkling phenomenon of the facings of sandwich materials has been treated by various authors as a stability problem of thin sheets on an elastic foundation. They have all made the usual assumptions that every component of the sandwich is homogeneous and perfect in its geometric configuration and that the core material possesses a finite stiffness but infinite strength. Consequently, their studies concern only classical buckling and yield the upper limit of the wrinkling stress. A brief review of their conclusions is given in the following.

Gough, Elam, and deBruyne[1] and Williams, Leggett, and Hopkins[2] treated with different degrees of rigor the case of a practical sandwich—i.e., a sandwich with a core of finite thickness with facings on both sides and complete adhesion between the faces and the core. The wrinkling stress of the facings was given in terms of a critical wave length, the properties of the facing material, and the Young's modulus of the core material. Their analyses were mainly concerned with sandwiches of metal facings and onazote or plywood core. Balsa core was discussed by the second group of authors, but it was approximated by "rather artificial materials . . . . that only resemble the true materials insofar as their isotropic qualities allow." They concluded that, for current designs, sandwich panels had no advantage over the more conventional sheet-stringer construction. The statement is believed to be due primarily to the fact that the balsa core has been treated as an isotropic material with a density of 10 lbs. per cu.ft., a shear modulus of 20,000 lbs. per sq.in., and Poisson's ratio of 0.3. The Young's modulus of this arbitrary material has been calculated as 52,000 lbs. per sq.in. Actually, for balsa of this density the with-grain modulus is 350,000 lbs. per sq.in., while the cross-grain modulus is 20,000 lbs. per sq.in. It will be shown later that the Young's modulus of the core in the direction perpendicular to the faces is critical and that considerable improvement in the wrinkling stress can be anticipated if the core is arranged with the grain normal to the faces, thus taking advantage of the higher modulus in the with-grain direction.

Hoff and Mautner[3] have recently reported work done on sandwiches of Papreg faces and cellular cellulose acetate core. They concluded that the wrinkling stress was independent of the geometric configuration of the sandwich and dependent only upon the moduli of the materials for the facings and the core. The following formula was recommended for a conservative estimate of the wrinkling stress:

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$$\sigma_{cr.} = {}^1/_2\sqrt[3]{E_f E_c G_c}$$

where

$\sigma_{cr.}$ = wrinkling stress in the facing
E_f = Young's modulus of the facing
E_c = Young's modulus of the core in the direction perpendicular to the facing
G_c = shear modulus of the core

If the proper values are substituted in the above formula, the wrinkling stress for sandwiches of aluminum-alloy faces and 10 lbs. per cu.ft. end-grain balsa ($E_f \approx$ 10,000,000 lbs. per sq.in., $E_c \approx$ 350,000 lbs. per sq.in., $G_c \approx$ 20,000 lbs. per sq.in.) will be 212,000 lbs. per sq. in. Even when a reasonable reduced modulus is introduced, the calculated wrinkling stress is still well above that observed in practice.

It is apparent that further study of theories concerning the wrinkling stress of sandwich materials has to be made so that a better understanding of the strength, as well as the stiffness, requirements of the core can be acquired and a closer correlation be obtained between the predicted strengths and those actually observed in test. In order to realize fully the high wrinkling strength obtainable in sandwich materials, it is desirable to have a core with a high strength and stiffness perpendicular to the facings, since the lateral restraint on the facings is due primarily to the core properties in that direction. Inasmuch as with wood and many of the synthetic core materials properties can be improved in one direction at the expense of those in the other two, sandwiches with aelotropic cores are discussed in detail under the section of "Analysis and Results," while the results for isotropic cores are summarized in the Appendix. Wrinkling of the facings in the plastic range or the effect of variable modulus is considered so that a solution can be obtained for any practical face material including metal, Fiberglas, Papreg, etc., provided the stress-strain relations are known. The Kármán reduced modulus is used in this connection. Although this reduced modulus applies only to struts with no initial curvature, it is believed that it offers the best means to express the plastic behavior of the facing material.

It is the contention of the author that the wrinkling stress of the facings is closely related to the initial waviness present in the facings. It is a well-known fact that curved struts possess no definite buckling load, and the ultimate load they can sustain depends somewhat on the yield strength of the material. Sandwich materials with initial waviness in the facings exhibit similar phenomenon. Most types of facings, especially metal, are made from commercial sheet stock, and as such they are usually not truly flat. Neither is the core material perfectly uniform in thickness, nor the material used for bonding. These initial irregularities may be corrected in part during the fabrication process, for example, by using stiff platens in the forming press, but the initial stresses set up in the bonding materials act in much the same way as initial waviness when the sandwiches are out of the press. It is apparent that initial waviness is present and might plausibly be expected to play a part in the wrinkling of the facings.

The physical picture of the initial waviness can be further clarified by the following considerations. Let the facings of a sandwich be loaded in compression. Any wave that might be present in the facings in the direction of the load will be amplified. This tends to set up tensile forces in the bond because the facings are restrained by the core from bowing out (or peeling off). This amplification will become greater as the load is increased, and there will soon be a time when the ultimate tensile strength of the bond (or the core) is exceeded. The facings will then separate from the core, and the failure of the complete sandwich occurs. Similar reasoning leads to the possibility of wrinkling of the facings because of failure of the core in compression or in shear rather than in tension. The theoretical wrinkling stress given by previous authors[1-3] may not be realized in all cases.

The amount of initial waviness present in the facings can vary over a wide range. However, it is not impossible to derive a rational expression to represent its general characteristics. It seems logical to assume that the amplitude of the initial waviness is greater for longer wave lengths and for thinner sheet thickness. The initial wave length can be taken to be the same as the wave length of the wrinkle in the facings, since at wrinkling (or failure) this is amplified to the greatest extent. The following expression, which has been introduced by Donnell[4] in the study of buckling of thin cylinders under axial compression, will be used in the present study:

$$W_0 = K\lambda^2/4\pi^2 t_f \qquad (1)$$

where

W_0 = amplitude of the initial waviness, in.
λ = wave length of the initial waviness, in.
t_f = thickness of the facings, in.
K = numerical constant—a measure of the waviness

It can easily be seen that this expression satisfies the characteristics described above. It has the additional advantage that any local weakness of bond or core can be incorporated in the empirical constant K when the wrinkling stress is to be evaluated.

Analysis and Results

Studies have been made on sandwiches of identical facings with aelotropic or isotropic cores. The results from rigorous treatment of boundary conditions are extremely cumbersome for practical application. An approximate solution has been found for the case of a sandwich with end-grain balsa core which makes possible the construction of a simple chart showing the re-

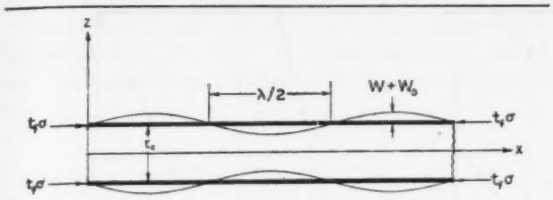

FIG. 1. Forces acting on a typical sandwich under compression.

lations between the various parameters associated with the wrinkling stress of the facings of a sandwich. This chart (Fig. 1) can be used for any core material exhibiting properties similar to end-grain balsa. This approximate solution is presented in this section to illustrate the effect of the initial waviness, while the more rigorous treatments are summarized in the Appendix.

It is assumed that the core material under consideration has a predominant Young's modulus in the direction perpendicular to the facings. The shear modulus and the Young's modulus in the transverse direction are considered to be zero. This is practically the case for end-grain balsa where the shear modulus and the Young's modulus in the transverse direction are, respectively, about one-twentieth and one-sixteenth of the Young's modulus in the longitudinal direction. The sandwich is assumed to wrinkle in the symmetrical attitude with a tension failure in the core. This assumption is based on numerous tests on sandwiches with aluminum-alloy faces and end-grain balsa core. It has been confirmed by comparing with the rigorous solution that, for balsa core, tension failure in the core occurs at practically the same facing stress for both the symmetrical and the antisymmetrical attitude, while shear failure in the core corresponds to a much higher facing stress.

The facings have the initial shape of

$$w_0 = W_0 \sin \alpha x \tag{2}$$

where

$W_0 = K\lambda^2/4\pi^2 t_f$, [Eq. (1)]
$\alpha = 2\pi/\lambda$
λ = wave length of wrinkle

When the facings are loaded in compression, the additional deflection shape can be taken as

$$w = W \sin \alpha x \tag{3}$$

The strain energy stored in a sandwich strip of unit width and one wave length is

$$V = 2\left[\frac{\tau E_f t_f^3}{24(1-\mu_f^2)}\right]\int_0^\lambda \left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{E_c}{2t_c}\int_0^\lambda (2w)^2 dx = \frac{1}{2}\lambda\left[\frac{\tau E_f t_f^3}{12(1-\mu_f^2)}\alpha^4 + \frac{2E_c}{t_c}\right] W^2$$

where

t_f = thickness of facing, in.
t_c = thickness of core, in.
τE_f = reduced modulus of facing material, lbs. per sq.in.
μ_f = Poisson's ratio of facing material
E_c = modulus of core-bond combination, lbs. per sq.in.

The virtual work done by the external forces (σt_f) due to a virtual displacement $\delta w = \delta W \sin \alpha x$ is

$$\delta T = 2\left(\frac{t_f\sigma}{2}\right)\int_0^\lambda \left\{\left[\frac{d}{dx}(w + w_0 + \delta w)\right]^2 - \left[\frac{d}{dx}(w + w_0)\right]^2\right\} dx = \lambda\sigma t_f \alpha^2 (W + W_0)\delta W$$

At equilibrium the virtual work is equal to the change of the strain energy caused by the virtual displacements, and it shows that

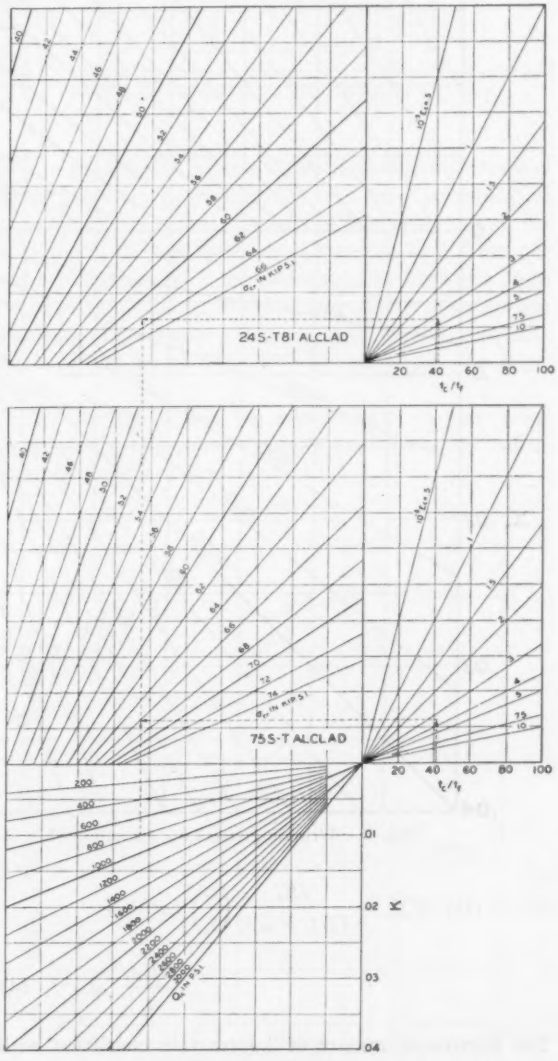

FIG. 2. Wrinkling stress $\sigma_{cr.}$ for sandwiches of end-grain balsa and 24S-T81 Alclad or 75ST Alclad facings.

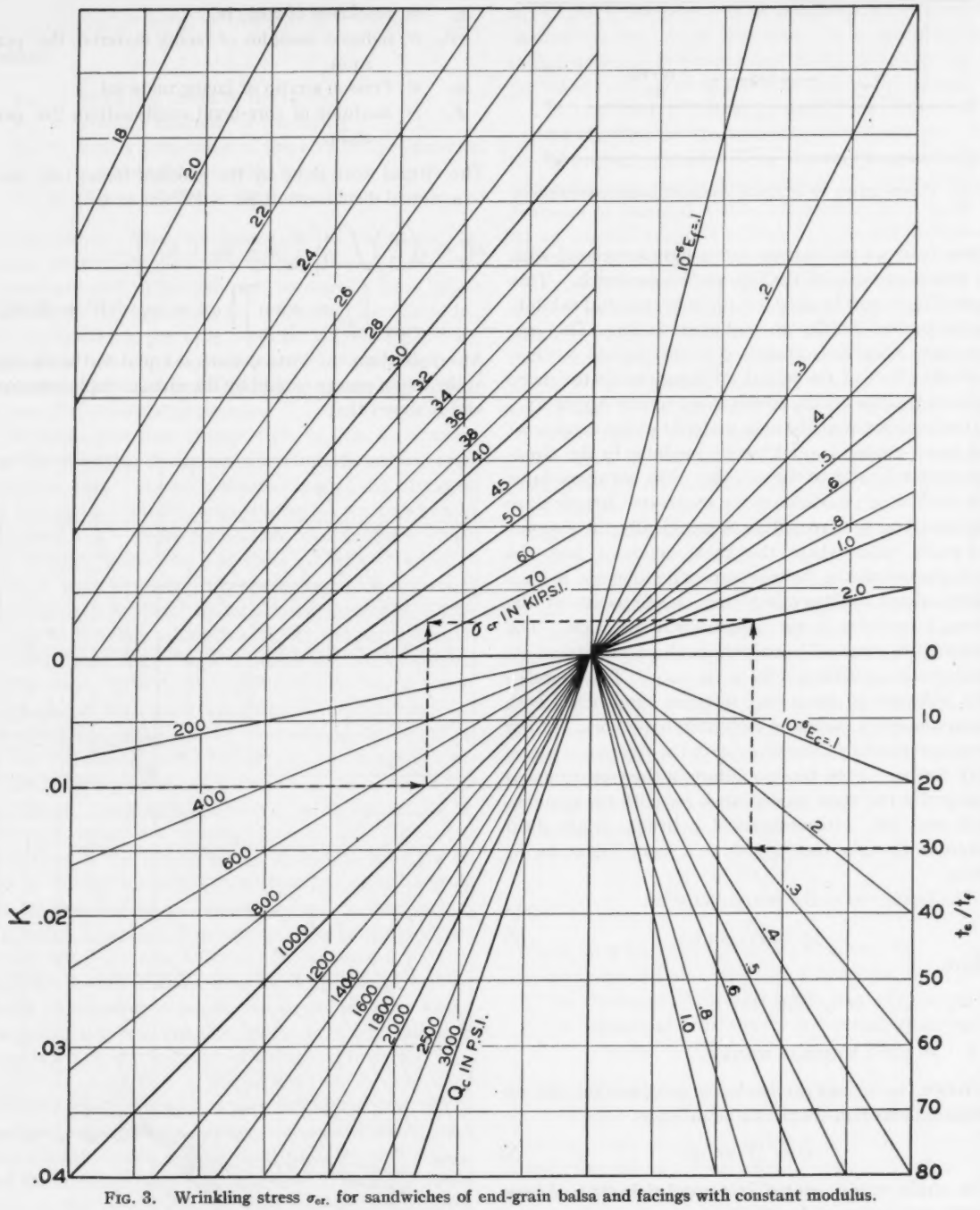

FIG. 3. Wrinkling stress $\sigma_{cr.}$ for sandwiches of end-grain balsa and facings with constant modulus.

$$\sigma[1 + (W_0/W)] = \frac{\tau E_f}{12(1 - \mu_f^2)}\left(\frac{t_f}{t_c}\right)^2(\alpha t_c)^2 + \frac{2E_c t_c}{t_c}(\alpha t_c)^{-2} \quad (4)$$

The maximum amount of deformation consistent with the ultimate tensile strength Q_c of the core-bond combination is

$$W = Q_c t_c/2E_c \quad (5)$$

The facing stress is given by the following expression after Eqs. (1) and (5) are substituted into Eq. (4):

$$\sigma = \frac{\frac{\tau E_f}{12(1 - \mu_f^2)}\left(\frac{t_f}{t_c}\right)^2(\alpha t_c)^2 + \frac{2E_c t_c}{t_f}(\alpha t_c)^{-2}}{(1 + \frac{K}{Q_c}\frac{2E_c t_c}{t_f}\alpha t_c)^{-2}} \quad (6)$$

The practical wrinkling stress is the minimum value of σ as given by Eq. (6). The minimization is carried out with respect to αt_c, and the wrinkling stress is given by

$$\sigma_{cr.} = \sigma_0(\sqrt{1 + \xi^2} - \xi) \tag{7}$$

where

$$\sigma_0 = \sqrt{2\tau E_f E_c t_f / 3(1 - \mu_f^2) t_c}$$
$$\xi = K\sigma_0 / 2Q_c$$

The wave length of the wrinkle at failure is accordingly

$$\lambda_{cr.} = \pi t_f \sqrt{2\tau E_f / 3(1 - \mu_f^2)\sigma_{cr.}} \tag{8}$$

It is to be noted that the wrinkling stress decreases with an increasing value of K. When K is set equal to zero, the wrinkling stress given by Eq. (8) is comparable to the theoretical values found by previous authors.[1, 2]

Eq. (7) can be given in the form of charts such as shown in Figs. 2 and 3. It is possible to construct charts for any material by noting that the intercepts are $1/\sigma_{cr.}$ on the K/Q_c axis and $2\tau E_f/3(1 - \mu_f^2)\sigma_{cr.}^2$ on the $t_c/E_c t_f$ axis. These results are readily obtained from rearranging Eq. (7) in polar form.

The charts given in Figs. 2 and 3 for 24S-T81 Alclad and 75ST Alclad faces are good only for tension or compression failure in the core or the bond. However, it will be shown in the Appendix that shear failures in the core are not probable if the ultimate shear strength of the core is greater than 12 per cent of its ultimate tensile or compressive strength. This implies that, when new core materials are under consideration, the ultimate tensile or compressive strength perpendicular to the faces should be considered to be of greater significance than the ultimate shear strength in developing high wrinkling stresses in the faces. It is realized, of course, that in any practical sandwich panel subjected to transverse loading, a high shear strength in the core is highly desirable.

It can be readily seen from the charts (Figs. 2, 3, and 4) that the faces of a sandwich can be stabilized to a higher stress by using a core of higher stiffness. It also indicates that for the same amount of initial imperfection the wrinkling stress can be increased by using core material of higher ultimate strength.

Application of Results—Value of K

The results of the present study are given in the form of charts in Fig. 2 for facings of 24S-T81 Alclad and 75ST Alclad sheets and core of end-grain balsa. Results for facings with constant modulus and end-grain balsa are given in Fig. 3. The case of sandwiches with isotropic core and facings of constant modulus and t_c/t_f of 40 is shown in Fig. 4. The application of these charts is best illustrated by an example.

Suppose that a sandwich specimen is made of two 0.012 gage 24S-T81 Alclad sheets and a 0.50-in. core of end-grain balsa, whose ultimate strength in tension is 800 lbs. per sq.in. and whose Young's modulus is 330,000 lbs. per sq.in. It follows from Fig. 2 that, for $K = 0.010$, the wrinkling stress would be 62,500 lbs. per sq.in. Conversely, if the facings are to sustain a compressive stress of 62,500 lbs. per sq.in., the ultimate tensile and/or compressive strength of the core material should be at least 800 lbs. per sq.in. for $K = 0.010$.

The amplitude and the wave length of the initial waviness for the above example can be readily calculated by means of Eqs. (8) and (1).

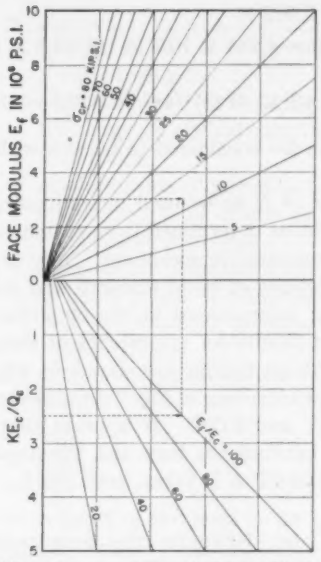

FIG. 4. Wrinkling stress $\sigma_{cr.}$ for sandwiches of isotropic core and facings with constant modulus when $t_c/t_f = 40$.

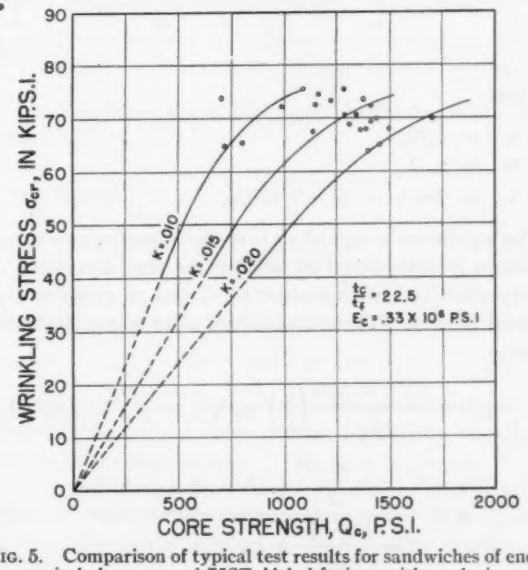

FIG. 5. Comparison of typical test results for sandwiches of end-grain balsa core and 75ST Alclad facings with analysis.

$$\lambda_{cr.} = \pi \times 0.012 \times \sqrt{0.76 \times 0.284 \times 10.5 \times 10^6/62.5 \times 10^3} = 0.23 \text{ in.}$$

$$W_0 = 0.010 \times (0.23)^2/4\pi^2 \times 0.012 = 0.0011 \text{ in.}$$

These values are considered to be well within expectations.

The value of K to be used in conjunction with the charts should be determined from statistical survey of tests on representative specimens. Any possible effect due to variations of bond strength and core strength can then be incorporated in the empirical constants. Typical test results for sandwiches of end-grain balsa and 75ST Alclad facings are shown in Fig. 5. Wrinkling stresses according to Fig. 2 are also shown for $K = 0.010$, 0.015, and 0.020. It appears that reasonable correlation between the tests and the present analysis can be obtained if K is taken to be 0.015.

Extension of Results to Sandwiches Under Biaxial Stresses

The approximate solution given previously for sandwiches under simple uniform compression can be extended to sandwiches under biaxial stresses. Let x and y denote the directions of the principal stresses and assume that σ_x is greater than σ_y. The wave length of the wrinkles in the x- and y-directions will be denoted by λ_x and λ_y. The expression for W_0 is assumed to be given by the following expression:

$$W_0 = K\lambda_x\lambda_y/4\pi^2 t_f \tag{9}$$

At equilibrium, the expression for the principal stress σ_x is

$$\sigma_x = \frac{1+m^2}{n+m^2}\left[\frac{\frac{\tau E_f}{12(1-\mu_f^2)}\left(\frac{t_f}{t_c}\right)^2 \gamma_x^2 + \frac{2E_c t_c}{t_f}\gamma_x^{-2}}{1 + \frac{K}{Q_c}\frac{2E_c t_c}{t_f}\frac{1+m^2}{m}\gamma_x^{-2}}\right]. \tag{10}$$

where

$$n = \sigma_y/\sigma_x$$
$$m = \lambda_y/\lambda_x$$
$$\gamma_x = 2\pi t_c\sqrt{(1+m^2)/m^2}/\lambda_x$$

The minimum value of σ_x corresponding to any stress ratio n is determined by setting the first derivative of expression (10) with respect to γ_x and m, respectively, equal to zero. These conditions, after rearrangements, are:

$$\gamma_x^2 = \frac{12(1-\mu_f^2)}{\tau E_f}\left(\frac{t_c}{t_f}\right)^2 \sigma_0(\sqrt{1+\xi_m^2} - \xi_m) \tag{11}$$

$$n = \frac{1 + \frac{1+m^2}{2}\xi_m(\sqrt{1+\xi_m^2} + \xi_m)}{1 + \frac{1+m^2}{2m^2}\xi_m(\sqrt{1+\xi_m^2} + \xi_m)} \tag{12}$$

where

$$\sigma_0 = \sqrt{2\tau E_f E_c t_f/3(1-\mu_f^2)t_c}$$
$$\xi_m = \frac{1+m^2}{m}\frac{K\sigma_0}{2Q_c} = \frac{1+m^2}{m}\xi$$

The critical wrinkling stress and the wrinkling wave length are, respectively,

$$\sigma_{x,\,cr.} = \frac{1+m^2}{n+m^2}\sigma_0(\sqrt{1+\xi_m^2} - \xi_m) \tag{13}$$

$$\lambda_{x,\,cr.} = \pi t_f\sqrt{\frac{(1+m^2)^2}{m^2(n+m^2)}\frac{\tau E_f}{3(1-\mu_f^2)\sigma_{x,cr.}}} \tag{14}$$

Solution for sandwiches under biaxial stresses can be found only by successive approximation. The presence of the reduced modulus has introduced a great deal of complication to the trial-and-error process. Since the above analysis is a crude approximate one, it seems rather unjustified to use its results for practical application. Therefore, the following alternative is suggested.

For equal compression in both directions ($n = 1$), m must be equal to unity from consideration of symmetry. This set of values satisfies Eq. (12). From Eqs. (13) and (14), the critical wrinkling stress and wave length are, accordingly,

$$\sigma_{cr.} = \sigma_x = \sigma_y = \sigma_0(\sqrt{1+4\xi^2} - 2\xi) \tag{15}$$

$$\lambda_{cr.} = \lambda_x = \lambda_y = \pi t_f\sqrt{2\tau E_f/3(1-\mu_f^2)\sigma_{cr.}} \tag{16}$$

It can be seen by comparing Eqs. (15) and (7) that the effect of equal biaxial compression is equivalent to doubling the value of the constant K. This indicates the possibility of applying a correction factor to K for biaxial stress conditions. The curve shown in Fig. 6 is believed to be on the conservative side for practical use. When the stress ratio is positive, the correction factor is assumed to vary linearly between the known

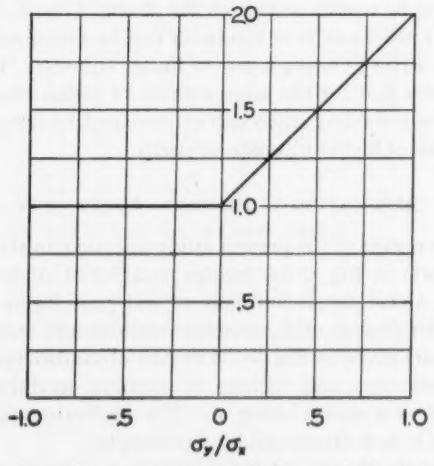

Fig. 6. Correction factor to be applied to K for the determination of $\sigma_{x,\,cr.}$ for biaxial loading.

end conditions. When the stress ratio is negative, the stiffening effect of the tensile stress is neglected.

Conclusion

An attempt has been made to study the strength requirements of core materials in sandwich construction. From consideration of the effect of initial waviness in the facings of sandwiches, a rational relation has been established between the ultimate core strength (or bond strength), the core modulus, and the wrinkling stress of the facings. A numerical constant K is introduced as a measure of the waviness of the facings. The magnitude of K for sandwiches with aluminum-alloy faces and end-grain balsa, as made with present production technique, has been found to lie between 0.01 and 0.02. Laboratory specimens might correspond to lower values of K.

Charts have been presented for wrinkling stress of sandwiches with 24S-T81 Alclad and 75ST Alclad facings and end-grain balsa core. These charts can also be used for establishing strength requirements for core materials.

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[3] Hoff and Mautner, *The Buckling of Sandwich Type Panels*, Journal of the Aeronautical Sciences, Vol. 12, No. 3, page 285, July, 1945.

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Appendix

A rigorous solution of the problem considered in the present paper is given in the following. The fundamental equations have been given by Williams, Leggett, and Hopkins.[2] The details of the derivation will be deleted for brevity. Necessary changes of notations have been made.

t_f = thickness of each facing
t_c = thickness of core
λ = wave length of wrinkle
$_\tau E_f$, μ_f = Young's modulus and Poisson's ratio of facing material
E_c, μ_c = Young's modulus and Poisson's ratio of isotropic core
E_c = Young's modulus of aelotropic core in the direction perpendicular to the facings
E_c' = Young's modulus of aelotropic core in the direction parallel to the facings
G_c = modulus of rigidity of aelotropic core
κ = E_c/E_c'
ν = E_c/G_c
μ_c = Poisson's ratio of aelotropic core associated with E_c
σ = compressive stress in facing
$\sigma_{cr.}$ = wrinkling (or critical compressive) stress in facing
u_c, w_c = displacements of points in the core
u_f, w_f = displacements of points in the middle surface of the facing
σ_x, σ_z, τ_{xz} = components of stress in the core
Q_c = ultimate tensile or compressive strength of the core-bond combination, whichever is lower
S_c = ultimate shear strength of the core
α = $2\pi/\lambda$
β = $\alpha t_c/2$
δ = E_f/E_c
ζ = $2t_f/t_c$
K = measure of waviness in the facing
W_0 = $K\lambda^2/4\pi^2 t_f$
w_0 = initial shape of assumed waviness in the facing = $W_0 \sin \alpha x$
q, s = values of σ_z and τ_{xz} at the common surface of the facing and the core

These equations are:

$$\frac{\tau E_f t_f^3}{12(1-\mu_f^2)}\frac{d^4 w_f}{dx^4} + t_f\sigma\frac{d^2(w_f+w_0)}{dx^2} + q = \frac{t_f}{2}\frac{ds}{dx} \quad \text{(A-1)}$$

$$S = E_f t_f (d^2 u_f/dx^2) \quad \text{(A-2)}$$

$$u_f + (t_f/2)\,(dw_f/dx) = [u_c]_{z=t_c/2} \quad \text{(A-3)}$$

$$w_f = [w_c]_{z=t_c/2} \quad \text{(A-4)}$$

and the stress-strain relations between σ_z, σ_x, τ_{xz} and u_c, w_c.

For isotropic cores, the stress distribution in the core can be described by stress function of the following type:

$$\phi = [A_1\alpha z \sinh \alpha z + A_2 \cosh \alpha z + B_1 \alpha z \cosh \alpha z + B_2 \sinh \alpha z] \sin \alpha x \quad \text{(A-5)}$$

For aelotropic cores, the stress function can be assumed as

$$\phi = \left[A_1 \cosh \frac{\alpha}{\alpha_1} z + A_2 \cosh \frac{\alpha}{\alpha_2} z + B_1 \sinh \frac{\alpha}{\alpha_1} z + B_2 \sinh \frac{\alpha}{\alpha_2} z\right] \sin \alpha x \quad \text{(A-6)}$$

where α_1^2, α_2^2 are the roots of the equation

$$\Lambda^2 - (\nu - 2\mu_c)\Lambda + \kappa = 0 \quad \text{(A-7)}$$

Symmetrical and antisymmetrical attitudes of wrinkling can be treated, respectively, by setting the B_i's and the A_i's equal to zero.

All the pertinent expressions for u_f, w_f, q, s can be obtained by straightforward manipulations. Eqs. (A-1) and (A-2) are then reduced to the following system of equations:

$$\left.\begin{aligned} a_{11}\begin{pmatrix}A_1\\B_1\end{pmatrix} + a_{12}\begin{pmatrix}A_2\\B_2\end{pmatrix} &= \alpha^{-2}K\sigma \\ a_{21}\begin{pmatrix}A_1\\B_1\end{pmatrix} + a_{22}\begin{pmatrix}A_2\\B_2\end{pmatrix} &= 0 \end{aligned}\right\} \quad \text{(A-8)}$$

The values of A_i's and B_i's are determined, and they are introduced into the expressions for σ_z and τ_{zz}. The magnitudes of these stresses are easily calculated at the face and the centerline of the core. Equating these stresses to the corresponding ultimate strength of the core, the stresses in the facings at equilibrium are expressed by the following formulas:

$$\frac{\sigma}{E_c} = F(\beta, \tau)/\left[G(\beta, \tau) + \frac{KE_c}{Q_c}\,Q(\beta, \tau)\right] \quad \text{(A-9)}$$

for tension failure.

$$\frac{\sigma}{E_c} = F(\beta, \tau)/\left[G(\beta, \tau) + \frac{KE_c}{S_c}S(\beta, \tau)\right] \quad \text{(A-10)}$$

for shear failure.

The functions $F(\beta, \tau)$, $G(\beta, \tau)$, and $S(\beta, \tau)$ are summarized as follows:

(A) *Antisymmetrical Attitude*

(1) Isotropic Core:

$$F(\beta, \tau) = -\beta + \tanh\beta + \beta\tanh^2\beta + \tau\delta\left\{\left[\frac{4 - 3\mu_f^2}{6(1 - \mu_f^2)}\beta^3\zeta^3 + (1 + \mu_c)\beta^3\zeta^2\right] + (1 - \mu_c)\beta^2\zeta^2\tanh\beta + [2\beta\zeta - (1 + \mu_c)\beta^3\zeta^2]\tanh^2\beta\right\} + \tau^2\delta^2\left[\frac{(1 + \mu_c)^2}{12(1 - \mu_f^2)}\beta^5\zeta^4(1 - \tanh^2\beta) + \frac{(3 - \mu_c)(1 + \mu_c)}{12(1 - \mu_f^2)}\beta^4\zeta^4\tanh\beta\right]$$

$$G(\beta, \tau) = 2\beta\zeta + \tau\delta[(1 + \mu_c)^2\beta^3\zeta^2(1 - \tanh^2\beta) + (3 - \mu_c)(1 + \mu_c)\beta^2\zeta^2\tanh\beta]$$

$$Q(\beta, \tau) = -\beta + \tanh\beta + \beta\tanh^2\beta + \tau\delta\left\{\frac{(1 + \mu_c)}{2}\beta^3\zeta^2 + \frac{(1 - \mu_c)}{2}\beta^2\zeta^2\tanh\beta + \left[2\beta\zeta - \frac{(1 + \mu_c)}{2}\beta^3\zeta^2\right]\tanh^2\beta\right\}$$

$$S(\beta, \tau) = \operatorname{sech}\beta\left(\beta\tanh\beta + \tau\delta\left\{\beta^2\zeta^2 - (1 + \mu_c)\beta^2\zeta + \left[(1 - \mu_c)\beta\zeta - \frac{(1 + \mu_c)}{2}\beta^3\zeta^2\right]\tanh\beta\right\}\right)$$

(2) Aelotropic Core:

$$F(\beta, \tau) = -\frac{1}{\alpha_2}\tanh\frac{\beta}{\alpha_1} + \frac{1}{\alpha_1}\tanh\frac{\beta}{\alpha_2} + \tau\delta\left[\frac{4 - 3\mu_f^2}{12(1 - \mu_f^2)}\frac{\alpha_2^2 - \alpha_1^2}{\alpha_1\alpha_2}\beta^3\zeta^3 + \frac{(\alpha_2^2 + \mu_c)}{\alpha_2}\beta^2\zeta^2\tanh\frac{\beta}{\alpha_1} - \frac{(\alpha_1^2 + \mu_c)}{\alpha_1}\beta^2\zeta^2\tanh\frac{\beta}{\alpha_2} + (\alpha_2^2 - \alpha_1^2)\beta\zeta\tanh\frac{\beta}{\alpha_1}\tanh\frac{\beta}{\alpha_2}\right] + \tau^2\zeta^2\left[\frac{(\alpha_2^2 + \mu_c)^2}{12(1 - \mu_f^2)\alpha_2}\beta^4\zeta^4\tanh\frac{\beta}{\alpha_1} - \frac{(\alpha_1^2 + \mu_c)^2}{12(1 - \mu_f^2)\alpha_1}\beta^4\zeta^4\tanh\frac{\beta}{\alpha_2}\right]$$

$$G(\beta, \tau) = \frac{\alpha_2^2 - \alpha_1^2}{\alpha_1\alpha_2}\beta\zeta + \tau\delta\left[\frac{(\alpha_2^2 + \mu_c)^2}{\alpha_2}\beta^2\zeta^2\tanh\frac{\beta}{\alpha_1} - \frac{(\alpha_1^2 + \mu_c)^2}{\alpha_1}\beta^2\zeta^2\tanh\frac{\beta}{\alpha_2}\right]$$

$$Q(\beta, \tau) = -\frac{1}{\alpha}\tanh\frac{\beta}{\alpha_1} + \frac{1}{\alpha_1}\tanh\frac{\beta}{\alpha_2} + \tau\delta\left[\frac{(\alpha_2^2 + \mu_c)}{2\alpha_2}\beta^2\zeta^2\tanh\frac{\beta}{\alpha_1} - \frac{(\alpha_1^2 + \mu_c)}{2\alpha_1}\beta^2\zeta\tanh\frac{\beta}{\alpha_1} + (\alpha_2^2 - \alpha_1^2)\beta\zeta\tanh\frac{\beta}{\alpha_1}\tanh\frac{\beta}{\alpha_2}\right]$$

$$S(\beta, \tau) = \frac{1}{\alpha_1\alpha_2}\left(\operatorname{sech}\frac{\beta}{\alpha_2} - \operatorname{sech}\frac{\beta}{\alpha_1}\right) + \tau\delta\left\{\frac{1}{2\alpha_1\alpha_2}\left[(\alpha_2^2 + \mu_c)\operatorname{sech}\frac{\beta}{\alpha_1} - (\alpha_1^2 + \mu_c)\operatorname{sech}\frac{\beta}{\alpha_2}\right]\beta^2\zeta^2 + \left[(\alpha_2^2 + \mu_c)\operatorname{sech}\frac{\beta}{\alpha_2}\tanh\frac{\beta}{\alpha_1} - (\alpha_1^2 + \mu_c)\operatorname{sech}\frac{\beta}{\alpha_1}\tanh\frac{\beta}{\alpha_2}\right]\beta\zeta\right\}$$

(B) *Symmetrical Attitude*

The functions $F(\beta, \tau)$, $G(\beta, \tau)$, and $Q(\beta, \tau)$ can be obtained by replacing "tanh" with "coth." The function $S(\beta, \tau)$ is not needed for the symmetrical attitude.

The critical compressive stress in the facing naturally has to be the minimum value of the Eqs. (A-9) and (A-10). This can be determined graphically in the following manner. Eqs. (A-9) and (A-10) are rear-

ranged as

$$\frac{1}{\sigma} = \frac{1}{E_c}\frac{G(\beta, \tau)}{F(\beta, \tau)} + \frac{K}{Q_c}\frac{Q(\beta, \tau)}{F(\beta, \tau)} \qquad \text{(A-9a)}$$

$$\frac{1}{\sigma} = \frac{1}{E_c}\frac{G(\beta, \tau)}{F(\beta, \tau)} + \frac{K}{S_c}\frac{S(\beta, \tau)}{F(\beta, \tau)} \qquad \text{(A-10a)}$$

Various values of β are then substituted into the above equations, and the resulting relations are plotted on a σ–τ diagram. The intersection with the σ–τ curve for the facing material locates the solution for σ corresponding to a particular value of β and K/Q_c or K/S_c. The minimum value of σ is obtained from a plot of σ versus β for various values of K/Q_c or K/S_c.

A comparison between the approximate solution given in the test and the rigorous solution presented above is shown in Fig. 7. It can be said that, for end-grain balsa, the tension failure will always happen first and that the approximate solution can be applied without any serious error.

This can also be deduced by comparing the function $Q(\beta, \tau)$ and $S(\beta, \tau)$. The minimum wrinkling stress corresponds to values of β greater than 3. For $\beta = 3$, $\coth \beta$ is practically the same as $\tanh \beta$, and $\operatorname{sech} \beta$ is of the order of $^1/_{10}$, and the results shown in Fig. 7 are to be expected.

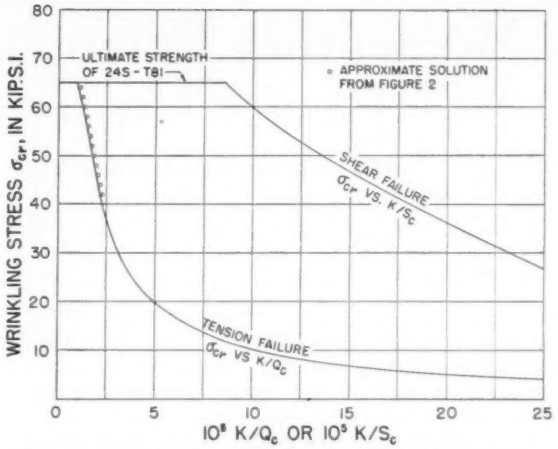

FIG. 7. Comparison of results for a sandwich of 0.012 in. 24S-T81 Alclad facings and 0.48 in. end-grain balsa (E_c = 250,000 lbs. per sq.in., μ_c = 0.50, κ = 16.4, ν = 22.6) from approximate and rigorous formulas.

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Momentum Relations in Propulsive Ducts*

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SUMMARY

This paper discusses the quantity

$$A(p + \rho V^2)$$

considered as a function of position along a steady "unidimensional" compressible flow through a duct. (A = area of cross section; p = pressure; ρ = density; V = velocity.) The function is called "stream thrust" and has the fundamental property that the difference between its values at any two sections of a stream gives the resultant of all normal and tangential forces acting on the duct walls between those sections. A description of the function is followed by some illustrations of its usefulness in connection with supersonic ram-jet design.

NOTATION

p	= pressure, lb. (mass) per sq.ft. absolute
ρ	= density, slugs per cu.ft.
k	= gas constant, ft.-lb. (force) per slug-°R.
T	= absolute temperature, °R.
γ	= ratio of specific heat at constant pressure to that at constant volume
A	= cross-sectional area, sq.ft.
V	= flow velocity, ft. per sec.
c	= sonic velocity, ft. per sec.
P	= stream thrust, defined by Eq. (3), lb. (force)
S_a	= air-specific impulse, defined by Eq. (11), lb. (force)-sec. per lb. (mass)
S_f	= fuel-specific impulse, defined by Eq. (11), lb. (force)-sec. per lb. (mass)
F	= mechanical force, lb. (force)
g	= acceleration of gravity, ft. per sec.^{2}
W	= total mass flow rate, lb. (mass) per sec.
a	= rate of mass flow of air, lb. (mass) per sec.
f	= rate of mass flow of fuel, lb. (mass) per sec.
M	= Mach Number ($= V/c$)
x	= defined by Eq. (15)
$\varphi(M)$	= defined by Eq. (6)
$\psi(x)$	= defined by Eq. (16)

Numerical subscripts refer to specific cross sections of a duct.

Literal subscripts have the following meanings:

s	= stagnation
c	= critical—i.e., when $M = 1$
bd	= burner drag

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INTRODUCTION

ANALYSIS OF THE OPERATION of a jet motor requires a new emphasis on momentum relations,[1] in contrast to the usual treatment of heat engines in which energy considerations predominate. This paper describes a working concept involving momentum which is applicable in any case of jet propulsion and which has proved useful in the practical design of ram-jet motors.

The next two sections formally define and describe this quantity, here called "stream thrust." The final section undertakes to illustrate its range of usefulness. The illustrations are drawn from a context of supersonic ram-jet analysis and design but do not in any way constitute a systematic treatment of this subject. They are intended only to exemplify an analytical approach, which is considered to have some novelty if this may ever be said of purely Newtonian mechanics. This point of view has been developed and used by the propulsion group at the Applied Physics Laboratory. For contributions to the material in its present form, particular acknowledgments are due to J. E. Cook and T. Davis.

DEFINITION OF STREAM THRUST

The process to be considered in this section is the steady-state flow of a gas through a duct, taking full account of its compressibility. For simplicity the following will be assumed:

(a) The gas obeys the usual equation of state

$$p/\rho = kT \tag{1}$$

and has a constant specific heat ratio γ.

(b) The duct wall is a surface of revolution.

(c) Velocity and other flow variables are uniform over each cross section of the duct, and transverse components of velocity are negligible. Skin friction and shocks may occur, however.

Under these circumstances, the resultant of all normal and tangential forces exerted by the flowing gas on the walls of the duct included between any two cross sections, (1) and (2) in Fig. 1, is axial in direction and has the magnitude F_{21} given by

$$F_{21} = A_2(p_2 + \rho_2 V_2^2) - A_1(p_1 + \rho_1 V_1^2)$$

or more compactly

$$F_{21} = P_2 - P_1 \tag{2}$$

where

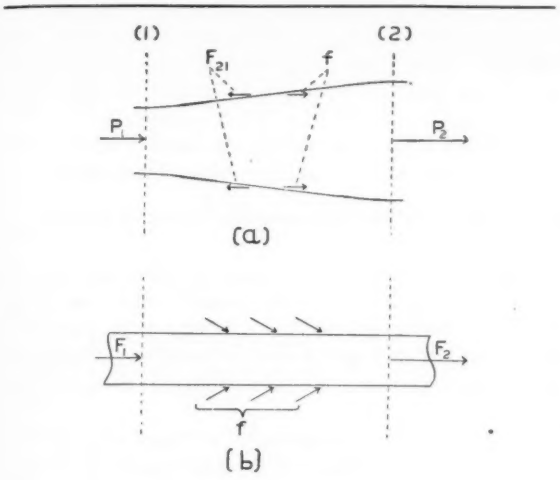

FIG. 1. Illustration of analogy between stream thrust in fluid flow and compressional forces in a rigid rod.
(a) Fluid flow through duct. P_1 and P_2 are entrance and exit stream thrusts. f is total force exerted by the duct walls on the stream. F_{21} is the force on the duct walls equal and opposite to f. See Eq. (2).
(b) Rigid rod. F_1 is force exerted on rod at one end. f is additional distributed force on rod. F_2 is force exerted by rod at the other end. F_1, F_2, and f are analogous to P_1, P_2, and f in (a).

$$P = A(p + \rho V^2) \qquad (3)$$

is regarded as a function of position along the stream, with values P_1 and P_2 at the corresponding sections (1) and (2). F_{21} is to be considered positive in Eq. (2) when directed from section (2) toward (1), regardless of the direction of stream flow.

Eq. (3) shows the function P to be made up of two parts: Ap equals the pressure force transmitted across a cross section, and $A\rho V^2$ equals the time rate of momentum flow across the same section. Eq. (2) is simply an expression of the fundamental mechanical principle that the time rate of creation of momentum within a region is equal to the resultant of all external forces acting across its boundaries. Considered in another way, Eq. (2) shows the force exerted by the stream on a duct wall to be measured directly by a change in the function P between the entrance and exit cross sections. This relation is sufficiently useful to warrant giving the quantity P a name. It will be called the "thrust of the stream" or "stream thrust," by its analogy to a compressive force or thrust transmitted along a rigid rod and caused to vary along the length of the rod by the existence of distributed external forces (see Fig. 1b). It should be apparent that the restrictions assumed above on the character of the flow are not essential and that a much more general definition of stream thrust is possible. Such a definition is stated in the Appendix.

The relation stated by Eq. (2) is generated by an impulse-momentum conservation law, which requires that the stream thrust be constant along the flow except as it is altered by external forces. Close parallels to this should be recognized in each of the following, possibly more familiar, relations.

(a) Conservation of mass leads to a mass-flow function, ρVA, which is constant along the stream except where matter is added or removed, as, for example, where fuel is injected into an air stream.

(b) Conservation of energy requires a constant rate of energy transport along the stream, symbolized by constant stagnation temperature T_s, except where this is altered by some energy source or sink, such as heat transfer across duct walls or combustion within a flowing gas.

(c) Along a stream with T_s constant, the second law of thermodynamics requires entropy also to be constant, symbolized by constant stagnation pressure p_s, except where this is altered by an irreversible process such as friction or shock.

Thus, stream thrust may be classed with mass flow, stagnation temperature, and stagnation pressure; these four parameters derive their importance from their connection with the four conservation laws cited.

DEPENDENCE OF STREAM THRUST ON W, T_s, AND M

Eq. (3) can easily be transformed to give the stream thrust P in terms of mass flow (W), stagnation temperature (T_s), and Mach Number (M). The result may be written

$$P = P_c\varphi(M) \qquad (4)$$

where

$$P_c = \frac{W}{g}\sqrt{\frac{2(\gamma + 1)}{\gamma}\,kT_s} = \frac{(\gamma + 1)Wc_c}{\gamma g} \qquad (5)$$

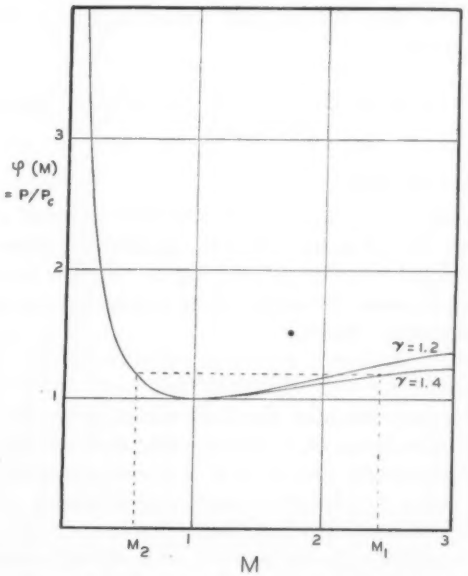

FIG. 2. Relative values of stream thrust as a function of Mach Number, for $\gamma = 1.2$ and 1.4. See Eqs. (4), (5), and (6) for definitions. M_1 and M_2 satisfy Eq. (7) and are a possible pair of Mach Numbers before and after normal shock in air.

and

$$\varphi(M) = \frac{1 + \gamma M^2}{\sqrt{2(\gamma + 1)}\, M[1 + {}^1/_2(\gamma - 1)M^2]^{1/2}} \quad (6)$$

$\varphi(M)$ is plotted in Fig. 2. It has a minimum at $M = 1$ and is normalized to unity at that point. This curve shows the relative variation of P with M along a stream in the common case in which W and T_s (and therefore P_c) are constant. Thus, one may consider the abscissa in Fig. 2 correlated with position along a stream flowing through a converging-diverging nozzle, from subsonic to supersonic velocity (or the reverse). The behavior of φ, or P, which first decreases and then increases, reflects the fact that the axial forces against the walls of the nozzle are in opposite directions on opposite sides of the throat. Some alternative forms of the function φ are shown in the Appendix.

P_c is the minimum value of the stream thrust, achieved when $M = 1$. Of the two expressions given for it in Eq. (5), the latter, involving the critical sonic speed c_c, is somewhat more basic, since it remains valid in the case where γ depends on temperature, provided γ is given the value appropriate to the critical state. Eq. (6) and the other part of Eq. (5) are not so readily adapted to use for variable γ.

Eq. (4) implies some interesting facts that may not otherwise be immediately apparent. For a stream along which P_c is constant (i.e., with mass flow and stagnation temperature constant), the force on the duct walls between any two sections depends only on the values of Mach Number at the two sections, regardless of the extent or character of shocks or other irreversibilities that may lie between. In particular, if entrance and exit Mach Numbers are equal, the net force on the intervening duct walls is zero, again regardless of irreversibilities.

Illustrative Applications of the Stream Thrust Function

(a) *Normal Shock*

The relation between Mach Numbers ahead of and behind a normal shock is readily obtained by means of the function φ in Eq. (6) and Fig. 2. Stream thrust, W, and T_s must, of course, all be continuous across a normal shock. Hence,

$$\varphi(M_1) = \varphi(M_2) \quad (7)$$

will give the relation of the Mach Numbers M_1 and M_2 before and after shock. Eq. (7) might be much simplified algebraically, but, if used in connection with the graph in Fig. 2, it affords a simple way of relating Mach Numbers across a shock, as illustrated in the figure.

In a similar way, the pressure ratio across a normal shock may be derived from a curve of ψ against x [see Eqs. (15) and (16) in the Appendix]. x_1 and x_2 must be related by

$$\psi(x_1) = \psi(x_2) \quad (8)$$

and

$$p_2/p_1 = x_1/x_2 \quad (9)$$

(since $A_1 = A_2$), where subscripts 1 and 2 again refer to points just ahead of and behind a normal shock.

(b) *Thrust Against the Expanding Section of an Exit Nozzle*

The thrust* of a jet motor may be calculated from a difference in stream thrust for exhaust and intake, as in Eq. (2). Taking the intake conditions to be constant, an additive change in the stream thrust at the exhaust will produce the same additive change in thrust of the motor.

If, for example, a nozzle with no expanding section (giving $M = 1$ at the exit) is to be compared with a deLaval nozzle with sufficient expansion (about 1.2:1) to give $M = 1.5$ at the exit, Fig. 2 [or Eq. (6)] shows that the gain in exit stream thrust is 6 per cent. The thrust of the motor [F_{21} in Eq. (2)] is increased by the same amount of added force but in a greater proportion, depending on the magnitude of the intake stream thrust [P_1 in Eq. (2)]. To complete a comparison of the two nozzles, the alteration in external or drag forces would, of course, also have to be taken into account.

(c) *Criterion of Quality of Combustor Performance*

It is sometimes desirable to state the thrust producing quality of an exhaust gas stream by means of an intensive property that is independent of the mass flow and also of the geometry of the exit nozzle. The stagnation temperature is a first approach to such a property. However, more satisfactory criteria, though not ideal, may be derived from the quantity P_c of Eq. (5). If P_c is understood to refer to the stream formed by combustion of fuel at rate f in air at rate a—i.e., with

$$W = a + f \quad (10)$$

then the quantities S_a and S_f defined by

$$a\, S_a = f\, S_f = P_c \quad (11)$$

are called, respectively, air-specific impulse and fuel-specific impulse. They are functions of the temperature and composition of the exhaust, hence depend on the ratio f/a and the completeness of combustion. S_a is of major significance when the design objective is maximum thrust from a limited air supply, as in the case of short-range flight of a ram-jet motor. S_f is of importance as a criterion of fuel economy, as in a long-range ram-jet. It is analogous to the term "specific impulse" as applied to rocket fuels.

* "Thrust" is used here and in section (d) as the total force on internal duct surfaces, based on absolute pressures. For practical purposes, only gage pressure should be used, so that thrust may vanish when the duct contains still air at one atmosphere. Shift to this latter basis can be made straightforwardly but, for simplicity, is omitted from the above discussions.

Since both S_a and S_f are derived directly from P_c, they are in the first instance performance criteria that characterize an exhaust stream with $M = 1$, separate knowledge of γ and c_c in Eq. (5) not being necessary. When an expanding nozzle is used to give $M > 1$ at the exit, the added thrust derived from the expansion depends on γ as well as P_c. However, as already pointed out, this added thrust is comparatively small for moderate expansions; therefore, in such cases, S_a and S_f are still important criteria of thrust producing quality, separate knowledge of γ being a secondary matter.

As well as being theoretically appropriate, S_a and S_f are rather directly determinable by experiments of two types:

(1) Direct thrust measurements made statically on a combustion chamber with exhaust at $M = 1$—i.e., with nonexpanding exit—operated with known rates of air and fuel flow. If air and fuel are introduced transversely, P_1 in Eq. (2) becomes zero, and the measured thrust directly determines the exit stream thrust P_c.

(2) Measurement of pressure at the inlet end of a cylindrical combustion chamber with the exhaust end unconstricted, as in Fig. 3. One must also know air and fuel flow rates and the total aerodynamic drag force within the chamber. The relations in this case are indicated more fully in the next section, Eq. (12).

(d) *Some Elementary Aspects of Ram-Jet Design*

A simple ram-jet motor is indicated schematically in Fig. 3. Section 2 divides the structure into its two principal parts: the forward portion, the diffuser, which receives incoming air during flight, reduces its velocity relative to the duct, and thereby develops an increased pressure; and the after section, the combustor, in which fuel is injected and burned to produce thrust. In the case here considered, the pressure p_2 at the diffuser exit is supposed sufficiently high to produce, at the combustor exit, sonic velocity ($M_3 = 1$) at a pressure above atmospheric (though necessarily much below p_2).

Suppose that specific values have been chosen for the flight speed and the maximum cross section A_2 (so that the aerodynamic drag of the motor may also be considered roughly fixed). One may then proceed to some rough design considerations.

The propulsive thrust* of the motor is, by Eq. (2), a difference between stream thrust at the exhaust and at the inlet. The latter will be determined by the rate of air intake, which for maximum thrust should, of course, be as small as allowed by other considerations. The exhaust stream thrust may be expressed as the stream thrust at section 2, diminished by the drag force existing beyond that point,

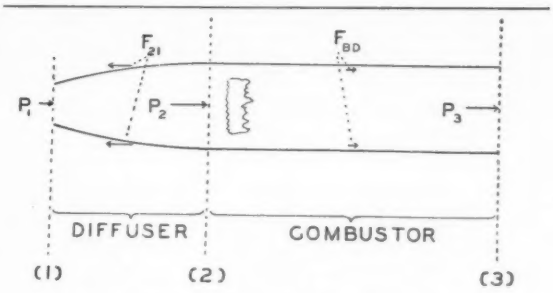

FIG. 3. Schematic diagram of a simple ram-jet motor.

$$P_3 = A_2(p_2 + \rho_2 V_2^2) - F_{bd} \tag{12}$$

Typically, the first of the three terms in the right member, $A_2 p_2$, is larger by an order of magnitude than either of the following two and thus roughly determines the available stream thrust P_3. We may then say that p_2—and, hence, P_3 approximately—is fixed by the flight velocity and diffuser efficiency. The P_3 so fixed is an upper limit to the possible thrust derivable from this motor.

On the other hand, P_3 must be related to the combustor performance by

$$P_3 = a\, S_a \tag{13}$$

This relation permits the choice of an air rate a consistent with available combustion quality S_a and the desired stream thrust P_3 allowed by the diffuser. There is an upper limit to air rates at which the pressure recovery in the diffuser is reasonably complete; Eq. (13) fixes a corresponding lower limit for a suitable S_a. If available values of S_a exceed this limit, the motor geometry indicated in Fig. 3 is feasible. If maximum thrust is required, the largest S_a and smallest a will be chosen; small S_a and larger a may give better fuel economy. The intake area A_1 is adjusted to be consistent with the desired air rate. Many other design questions can be treated by similar considerations.

APPENDIX

(a) *General Definition of Stream Thrust*

A generalized definition of a stream thrust vector can easily be written with the aid of dyadic notation. Let $\boldsymbol{S}$ be the stress dyadic, so defined that $\boldsymbol{S}\cdot d\boldsymbol{A}$ is the force transmitted from the negative to the positive side of the elementary area $d\boldsymbol{A}$ in the fluid (by pressure and viscosity). Then the stream thrust vector $\boldsymbol{P}$ is defined as

$$\boldsymbol{P} = \int_A (\boldsymbol{S} + \rho \boldsymbol{V}\boldsymbol{V})\cdot d\boldsymbol{A}$$

The integral is taken over an arbitrary entrance or exit area $\boldsymbol{A}$, with $d\boldsymbol{A}$ in the downstream direction. Eq. (2) may then be used as a vector equation for a duct of any shape.

* See footnote, page 542.

(b) *Alternative Forms for $\varphi(M)$*

The simplest mathematical form is assumed by the function $\varphi(M)$ defined by Eq. (6) if it is expressed in terms of the ratio (V/c_c). The result is

$$\varphi(M) = [1 + (V/c_c)^2]/2(V/c_c) \tag{14}$$

It is sometimes useful to have the function $\varphi(M)$ expressed in terms of still another independent variable—namely, the product of pressure and area of cross section. A dimensionless parameter x involving this product is defined by

$$x = \frac{P_c}{(\gamma+1)Ap} = \sqrt{\frac{2}{\gamma+1}}\, M\left[1 + \frac{1}{2}(\gamma-1)M^2\right]^{1/2} \tag{15}$$

The function $\varphi(M)$ can also be expressed in terms of x:

$$\psi(x) = \varphi(M) = \frac{\gamma\sqrt{1 + (\gamma^2-1)x^2} - 1}{(\gamma^2-1)x} \tag{16}$$

A curve of $\psi(x)$ plotted against x would be somewhat similar to Fig. 2, since x increases monotonically with M and is equal to M when both are equal to 1.

REFERENCE

[1] See, for example, Moss, Sanford A., and Foote, W. R., *Some Elementary Details of Ramming Intake and Propulsive Jets for Airplanes*, Journal of the Aeronautical Sciences, Vol. 13, No. 3, pp. 111–118, March, 1946 (a discussion of momentum relations for incompressible flow).

Errata

The following corrections to the article "Stress Analysis of Rotor Blades" (JOURNAL, Vol. 14, No. 6, p. 315, June 1947) were submitted by the author, Gabriel Horvay.

(1) Page 315.—Eqs. (39a) and (39b) replace φ by ψ.

(2) Page 317.—Under heading "Outline of the Tabular Procedure....." change third line to: "....; inflow velocity and lift curve slope are constant through the rotor disc; flapping hinge is uncocked and coincides with the center of rotation, higher than second harmonic effects can be neglected," etc.

(3) Page 319.—In Eq. (14d), subscript k belongs outside of the brace.

(4) Page 320.—Eq. (18) should read:

$$\{\delta L_{st.}\}_k = q_k\{\lambda x + \vartheta x^2\}_k$$

(5) Page 321.—Eq. (24) should read:

$$\delta J = 0 \text{ etc.)}$$

(6) Page 324.—Eq. (42) should read:

$$\varphi = \Upsilon/x = \beta = \ldots.$$

(7) Page 326.—The last line in the right column should read: "and the numerical columns $\cos 0$, $\cos 2\psi$, $\sin 2\psi$ of Table 4b...."

(8) Page 330.—In Eq. (44*), integrate from zero.

(9) Page 330.—In the right column, the seventh line from the top should read: "using the deflection curve $y = 1 - \cos {}^1/_2\pi x$, and making...."

(10) Page 331.—Replace Eq. (71) by:

$$\{S_C\}_k = -\{C\varphi\}_k + \sum_k^{\text{tip}} \tau y/R$$

(11) Page 331.—Add the following at the end of the sentence following Eq. (71): In performing the calculations, it is convenient to replace the steps S_C and S_J by

$$S_1 = -\{C\varphi\}_k \tag{71a}$$

and

$$S_2 = \sum_k^{\text{tip}} (y - \ddot{y}/\omega^2)\tau/R \tag{71b}$$

Evidently,

$$S_1 + S_2 = S_C + S_J \tag{71'}$$

(12) Page 331.—In Eq. (75), double dot ζ in the first term.

(13) Page 331.—In Eq. (80), double dot y in the last term of the first member.

(14) Page 331.—In Eq. (81b), precede $\sin 2\psi$ by factor Υ_4.

(15) Page 331.—In Eq. (82b), replace $\ddot{\zeta}_n$ by $(\ddot{\zeta})_n$ and add footnote: e.g., $(\ddot{\zeta})_4 = 2\omega A_2$.

(16) Page 331.—In Eq. (83), add subscript n to Υ.

(17) Page 332.—In Eq. (84), replace $c(x)$ by $C(x)$.